Mathematical Methods (10/24.539)

IX. The Sturm-Liouville Problem and Generalized Fourier Series

Introduction

The special functions discussed in the previous section are mostly special cases of a particular class of problems called ***Sturm-Liouville Problems*** or simply ***Eigenvalue Problems***. Most homogeneous 2-point ***Boundary Value Problems*** (BVPs) with homogeneous boundary conditions (BCs) fall into this classification. The Sturm-Liouville problem is important because the solutions to a homogeneous BVP with homogeneous BCs produce a set of orthogonal functions -- and we have seen that these are important for applications via ***Generalized Fourier Series***. Thus, this section of notes is designed to generalize some of the material from Section VIII on Special Functions and Orthogonality and to give the student a good introduction to the base terminology and solution techniques associated with classical eigenvalue problems (continuous 2^{nd} order homogeneous BVPs with homogeneous boundary conditions) and the use of the resultant orthogonal functions within Generalized Fourier Series.

The topics covered in this section are itemized below:

Overview and General Terminology

As outlined above, many of the special functions we have discussed previously represent specific cases of a more generalized Sturm-Liouville problem. In particular, a 2-point boundary value problem having the form

$$\frac{d}{dx}\left[r(x)\frac{d}{dx}y(x)\right]+\left[q(x)+\lambda p(x)\right]y(x)=0 \tag{9.1}$$

on some given interval, $a \le x \le b$, with homogeneous boundary conditions given by

$$\begin{aligned} c_1 y(a)+c_2 y'(a)=0 \\ k_1 y(b)+k_2 y'(b)=0 \end{aligned} \tag{9.2}$$

where c_1, c_2, and k_1, k_2 are constants, the p(x), q(x), and r(x) coefficients are differentiable functions of the independent variable (with p(x) > 0), and λ is a parameter, is called a ***Sturm-Liouville Problem***.

Important Notes:

1. Homogeneous 2-point boundary value problems with homogeneous boundary conditions have an ***infinite number of solutions***.

2. The values of λ_n that give non-trivial solutions are referred to as ***eigenvalues*** and the corresponding solutions, $y_n(x)$, are ***eigenfunctions***.

3. The set of eigenfunctions, $\{y_n(x)\}$, form an ***orthogonal system with respect to the weight function, p(x)***, over the interval $a \le x \le b$.

4. If p(x), q(x), and r(x) are real, the eigenvalues are also real (see any good text on Advanced Engineering Mathematics and Problem 11.24 in the Schaum's Outline, ***Advanced Mathematics***, for a general proof).

It is the orthogonal eigenfunction solutions and their application within Generalized Fourier Series that make the Sturm-Liouville Problem so important.

Example 9.1 represents a good illustration of the solution techniques utilized for typical eigenvalue problems and it identifies, via example, much of the terminology from above. It is a simple demonstration that hopefully clarifies many of the basic concepts and notation associated with the so-called Sturm-Liouville Problem.

Example 9.2 then expands upon the base terminology by illustrating how to solve some simple BVPs using a Generalized Fourier Series expansion. It also emphasizes, once again, the important role that the endpoint conditions have in determining the completeness of the eigenfunctions in a particular application.

Example 9.1 -- A Simple Eigenvalue/Eigenfunction Problem

Problem Description:

Let's illustrate many of the concepts and statements associated with typical eigenvalue problems with an example that demonstrates several important items:

1. Analytical technique for finding eigenvalues and eigenfunctions
2. Normalization of the eigenfunctions
3. Orthogonality of eigenfunctions for this particular case
4. Method for expanding arbitrary functions in sets of orthonormal eigenfunctions -- an example of Generalized Fourier Series Eigenfunction expansions

In particular, consider the following homogeneous equation with homogeneous boundary conditions (this just happens to represent the physics of a vibrating elastic string):

$$y'' + \lambda y = 0$$

with

$$y(0) = 0 \quad \text{and} \quad y(L) = 0$$

Problem Solution:

I. Show that this is a special case of a Sturm-Liouville Problem:

We first let $r(x) = p(x) = 1$ and $q(x) = 0$. With these substitutions, eqn. (9.1) becomes

$$\frac{d}{dx}\left[r(x)\frac{d}{dx}y(x)\right] + \left[q(x) + \lambda p(x)\right]y(x) = 0$$

or

$$\frac{d}{dx}\left[(1)\frac{d}{dx}y(x)\right] + \left[(0) + \lambda(1)\right]y(x) = \frac{d^2}{dx^2}y(x) + \lambda y(x) = 0$$

which is the desired ODE for this problem.

Now with boundary points a = 0 and b = L, the coefficients in the general boundary condition equation [see eqn. (9.2)] become

$$c_1 = 1, \;\; c_2 = 0 \qquad \text{and} \qquad k_1 = 1, \;\; k_2 = 0$$

With these substitutions, the BCs in the general equation can be written simply as

$$y(0) = 0 \quad \text{and} \quad y(L) = 0$$

which again are the desired conditions for this problem. Thus, the current situation is indeed a special case of the general Sturm-Liouville problem.

II. Find the eigenvalues and eigenfunctions for this problem:

In trying to determine the solution to this problem, there are three possibilities -- the eigenvalues might be negative, zero, or positive (note that, since the coefficient functions are real, we already know that the eigenvalues will be real). Let's try each possibility:

Case 1 - Negative Eigenvalues: For this case we try $\lambda = -\nu^2$. With this substitution, the original ODE becomes

$$y'' - \nu^2 y = 0$$

This is just a simple, constant coefficient, second-order ODE with characteristic equation

$$r^2 = \nu^2$$

and roots

$$r_{1,2} = \pm\nu$$

Thus, the general solution for the negative eigenvalue assumption is

$$y(x) = A_1 e^{\nu x} + A_2 e^{-\nu x}$$

Applying the first boundary condition gives

$$y(0) = 0 = A_1 + A_2 \qquad \text{or} \qquad A_2 = -A_1$$

Similarly, the second boundary condition gives

$$y(L) = 0 = A_1 e^{\nu L} + A_2 e^{-\nu L} = A_1\left(e^{\nu L} - e^{-\nu L}\right) = 2A_1 \sinh \nu L$$

Therefore, for real $\nu > 0$, $A_1 = 0$ and $A_2 = 0$, and the only solution is the trivial solution, $y(x) = 0$. Thus, for this problem, letting λ be negative was ***not*** a good choice.

Case 2 - Zero Eigenvalues: For this case we try $\lambda = 0$. With this substitution, the original ODE reduces to $y'' = 0$. This can be integrated twice to give

$$y'(x) = c_1 \qquad \text{and} \qquad y(x) = c_1 x + c_2$$

Applying the first boundary condition gives

$$y(0) = 0 = c_1(0) + c_2 \qquad \text{or} \qquad c_2 = 0$$

Using the same logic, the second boundary condition gives

$$y(\pi) = 0 = c_1 \times L \qquad \text{or} \qquad c_1 = 0$$

Therefore, as for Case 1, the only solution for the case of zero eigenvalues is the trivial solution, $y(x) = 0$.

Case 3 - Positive Eigenvalues: Since the first two choices did not lead to valid solutions, we should be hopeful for success for this third case. As before, we try $\lambda = +\nu^2$, and substitution into the original ODE gives

$$y'' + \nu^2 y = 0$$

This time the characteristic equation is

$$r^2 = -\nu^2$$

with roots

$$r_{1,2} = \pm i\nu$$

With pure imaginary roots, the general solution can be written as

$$y(x) = A_1 e^{i\nu x} + A_2 e^{-i\nu x}$$

or, more commonly, one writes

$$y(x) = c_1 \cos \nu x + c_2 \sin \nu x$$

Using this latter expression and the boundary condition at x = 0 gives

$$y(0) = 0 = c_1 \times 1 + c_2 \times 0 \qquad \text{or} \qquad c_1 = 0$$

Therefore, the general solution reduces to

$$y(x) = c_2 \sin \nu x$$

Now using the boundary condition at x = L gives

$$y(L) = 0 = c_2 \sin \nu L$$

For a nontrivial solution [i.e. $c_2 \neq 0$], we must require that $\sin \nu L = 0$. Since we know that $\sin n\pi = 0$ for $n = 0, 1, 2, \cdots$, we require that $\nu L = n\pi$ for $n = 1, 2, 3, \cdots$, where n = 0 has been excluded since this leads to a trivial solution. Thus we see that the second boundary condition leads to a constraint equation that defines specific values of λ that give nontrivial solutions to the original ODE. This constraint equation is often referred to as an ***eigencondition***. Note that there are an infinite number of suitable values for λ, since n can be any positive integer.

Thus, the eigenvalues for this problem are $\nu = n\pi/L$ for $n = 1, 2, 3, \cdots$ or, in terms of the parameter, λ, in the original equation, we have $\lambda = \nu^2 = (n\pi/L)^2$ for $n = 1, 2, 3, \cdots$. Finally, the eigenfunction corresponding to the n^{th} eigenvalue is simply

$$y_n(x) = \sin n\pi x/L$$

Note that the c_2 coefficient in the above general solution has been set to unity for convenience (at this point). The second BC, instead of determining the second coefficient in the general solution, gave us the eigencondition that specifies the eigenvalues for this problem. This situation leaves c_2 undetermined. However, because the original ODE is homogeneous, any arbitrary normalization could be used. Here we choose the normalization to be unity -- but later, when trying to define an ***orthonormal*** set of functions, a new normalization is determined (see below).

III. Show that the eigenfunctions for this problem are orthogonal:

For this problem, the definition of orthogonality requires the following equality for $m \neq n$:

$$\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)dx = 0$$

From a set of integral tables (for $a^2 \neq b^2$), we have

$$\int_{x_0}^{x_1} \sin(ax)\sin(bx)dx = \frac{1}{2}\left[\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right]\Bigg|_{x_0}^{x_1}$$

and for our case, this becomes

$$\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)dx = \frac{1}{2}\left[\frac{\sin(m-n)\pi}{(m-n)\pi/L} - \frac{\sin(m+n)\pi}{(m+n)\pi/L} - 0 + 0\right]$$

However, $\sin p\pi = 0$ for all integer p (including negative values). Therefore,

$$\sin(m-n)\pi = 0 \qquad \text{and} \qquad \sin(m+n)\pi = 0$$

which proves the above expression.

IV. Find the set of normalized eigenfunctions [i.e. $\|g_n\| = 1$] for this problem:

To completely define the orthogonality condition, one needs to determine the value of the above integral when m = n. For the case of ***orthonormal eigenfunctions***, the solutions to the original ODE are normalized to give a value of unity for this integral. In particular, for the unnormalized eigenfunctions, $\sin n\pi x/L$, we have

$$\int_0^L \sin^2(n\pi x/L)dx = \frac{x}{2} - \frac{\sin 2n\pi x/L}{4n\pi/L}\Bigg|_0^L = \frac{L}{2}$$

Therefore, if we let

$$g_n(x) = \sqrt{\frac{2}{L}}\sin n\pi x/L$$

then an orthonormal set of eigenfunctions, $g_n(x)$, results. This is shown explicitly by the normalization expression,

$$\|g_n\|^2 = \frac{2}{L}\int_0^L \sin^2(n\pi x/L)\,dx = \left(\frac{2}{L}\right)\left(\frac{L}{2}\right) = 1$$

Clearly, for the normalized functions, the norm, $\|g_n\|$, is unity (as designed).

V. Finally, as an example of using a Fourier series expansion, let's expand a few simple functions, f(x), using the orthonormal basis functions defined above. With the Fourier series, the function f(x) can be written as

$$f(x) = \sum_{n=1}^{\infty} a_n g_n(x)$$

Case 1: The Constant Function

For example, if f(x) is unity over the interval [0,L], we have

$$f(x) = 1 = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n \sin n\pi x/L$$

where $\|g_n\| = 1$, since the $g_n(x)$ have already been normalized.

To find the expansion coefficients we multiply f(x) by $g_m(x) = \sqrt{\frac{2}{L}} \sin m\pi x/L$ and integrate over the domain of interest, giving

$$\int_0^L f(x) g_m(x) dx = \sqrt{\frac{2}{L}} \int_0^L (1) \sin m\pi x/L \, dx = \sum_{n=1}^{\infty} a_n \left(\frac{2}{L}\right) \int_0^L (1) \sin(m\pi x/L) \sin(n\pi x/L) dx$$

However, since the eigenfunctions are orthonormal functions, the RHS reduces to a_m because all the terms in the summation are zero except when n = m, and we have

$$a_m = \int_0^L f(x) g_m(x) dx = \sqrt{\frac{2}{L}} \int_0^L \sin m\pi x/L \, dx = \sqrt{\frac{2}{L}} \left[\frac{-\cos m\pi x/L}{m\pi/L}\right]_0^L = \sqrt{\frac{2}{L}} \left[\frac{1}{m\pi/L} - \frac{\cos m\pi}{m\pi/L}\right]$$

But $\cos m\pi = (-1)^m$, therefore,

$$a_m = \sqrt{\frac{2}{L}} \frac{L}{m\pi} \left(1 - (-1)^m\right) = \begin{cases} \sqrt{\frac{2}{L}} \frac{2L}{m\pi} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

and

$$f(x) = 1 = \sum_{n}^{\infty} \left[\sqrt{\frac{2}{L}} \frac{2L}{n\pi}\right] \left[\sqrt{\frac{2}{L}} \sin n\pi x/L\right] \quad \text{for } n = 1, 3, 5, \cdots$$

or

$$1 \approx \frac{4}{\pi} \sum_{n}^{N} \frac{\sin n\pi x/L}{n} \quad \text{for } n = 1, 3, 5, \cdots$$

Note also that this can be written as

$$1 \approx \frac{4}{\pi} \sum_{n}^{N} \frac{\sin\left((2n-1)\pi x/L\right)}{(2n-1)} \quad \text{for } n = 1, 2, 3, \cdots$$

where now the indexing is simply incremented by unity and only the nonzero terms are included.

The approximation given in the last expression has been written as a finite expansion, where N represents the number of terms used (this is often the number of nonzero terms). This finite expansion is evaluated for several different N in the Matlab file **eigenf1.m**, and the partial sums generated by these calculations are graphed in Fig. 9.1. The **eigenf1.m** file is listed below in Table 9.1.

This particular evaluation of the Fourier series is not very elegant or efficient, but it is easy to see exactly how the computations are performed. As the number of terms used in the expansion increases, we would expect to see better and better agreement with the desired function (a constant line at a numerical value of unity for this case). This is the general trend that is observed in Fig. 9.1 in the center of the desired interval but, at the end points, this can never occur because all the expansion functions are identically zero at the interval boundary points. Thus, in this case, we can never have an exact representation over the entire interval, even with an infinite number of terms in the expansion.

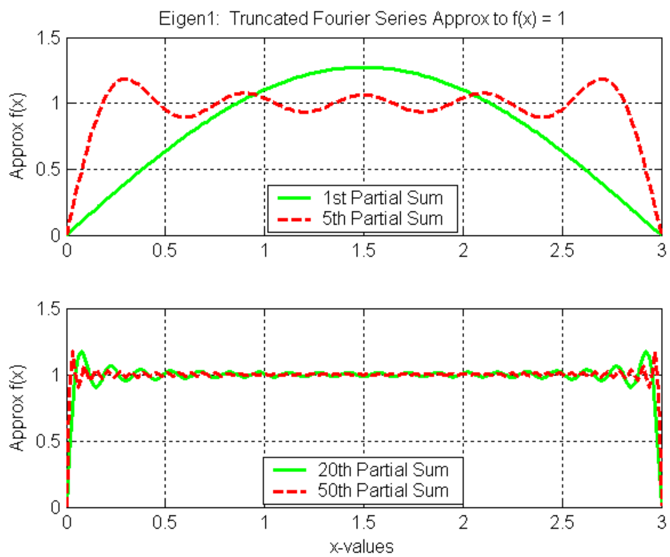

Fig. 9.1 Some partial sums for the Fourier series for constant f(x).

Table 9.1 Listing of Matlab file eigenf1.m.

```
%
%   EIGENF1.M   Demo of Fourier Series Representation for f(x) = 1
%
%   The goal here is to evaluate the infinite series expansion for
%   f(x) = 1 in terms of sinusoids (i.e. a Fourier series expansion)
%
%   For this problem, the coefficients and basis functions are (see notes):
%      an = (4/pi)/m         gn(x) = sin(m*pi*x/L)
%   with m = (2n-1)   and n = 1, 2, ...
%
%   Note:  This implementation is not very efficient or elegant, but it
%   is fairly straightforward.  A related example, EIGENF2.M, gives a more
%   efficient set of coding for evaluating infinite series.
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003))
%

%
%   getting started
      clear all, close all, nfig = 0;
%
```

```
%   set x domain and some constants
      L = 3;    x = linspace(0,L,201);    c = 4/pi;
%
%   calc 1st partial sum
      s1 = c*sin(pi*x/L);
%
%   calc 5th partial sum
      s5 = s1;
      for n = 2:5
        m = 2*n-1;   s5 = s5 + (c/m)*sin(m*pi*x/L);
      end
%
%   calc 20th partial sum
      s20 = s5;
      for n = 6:20
        m = 2*n-1;   s20 = s20 + (c/m)*sin(m*pi*x/L);
      end
%
%   calc 50th partial sum
      s50 = s20;
      for n = 21:50
        m = 2*n-1;   s50 = s50 + (c/m)*sin(m*pi*x/L);
      end
%
%   plot curves
      nfig = nfig+1;  figure(nfig)
      subplot(2,1,1),plot(x,s1,'g-',x,s5,'r--','LineWidth',2),grid
      title('Eigen1:  Truncated Fourier Series Approx to f(x) = 1')
      ylabel('Approx f(x)')
      legend('1st Partial Sum','5th Partial Sum')
%
      subplot(2,1,2),plot(x,s20,'g-',x,s50,'r--','LineWidth',2),grid
      xlabel('x-values'),ylabel('Approx f(x)')
      legend('20th Partial Sum','50th Partial Sum')
%
%   end of demo
```

Case 2: The Quadratic Function

As another example, let's assume that f(x) varies quadratically over the interval [0,L] with zero endpoint values. In particular, letting $f(x) = x(L-x)$, we have

$$f(x) = x(L-x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n \sin n\pi x/L$$

where again $\|g_n\| = 1$, since the $g_n(x)$ have already been normalized.

As before, to find the expansion coefficients we multiply f(x) by $g_m(x)$, integrate over the domain of interest, and use the fact that the eigenfunctions are orthonormal to give

$$a_m = \int_0^L f(x) g_m(x) dx = \sqrt{\frac{2}{L}} \int_0^L x(L-x) \sin m\pi x/L \, dx$$

$$= \int_0^L L\sqrt{\frac{2}{L}}\, x \sin m\pi x/L \, dx - \int_0^L \sqrt{\frac{2}{L}}\, x^2 \sin m\pi x/L \, dx$$

$$= L\sqrt{\frac{2}{L}} \left[\left(\frac{1}{(m\pi/L)^2} \sin m\pi x/L - \frac{1}{m\pi/L} x \cos m\pi x/L \right) \right]_0^L$$

$$- \sqrt{\frac{2}{L}} \left[\frac{2x}{(m\pi/L)^2} \sin m\pi x/L + \frac{2}{(m\pi/L)^3} \cos m\pi x/L - \frac{x^2}{m\pi/L} \cos m\pi x/L \right]_0^L$$

Using the equalities that $\sin m\pi = 0$ and $\cos m\pi = (-1)^m$ for integer m, evaluation of the above expressions at the endpoints gives

$$a_m = \sqrt{\frac{2}{L}}\left[L\left(-\frac{L}{m\pi/L}\right)(-1)^m - \left(\frac{2}{(m\pi/L)^3}(-1)^m - \frac{L^2}{m\pi/L}(-1)^m - \frac{2}{(m\pi/L)^3}\right)\right]$$
$$= \sqrt{\frac{2}{L}}\left[-\frac{L^3}{m\pi}(-1)^m + \frac{2L^3}{(m\pi)^3}\left(1-(-1)^m\right) + \frac{L^3}{m\pi}(-1)^m\right]$$
$$= \begin{cases} \sqrt{\frac{2}{L}}\frac{4L^3}{(m\pi)^3} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

Finally, the finite Fourier series representation can be written as

$$f(x) = x(L-x) \approx \frac{8L^2}{\pi^3}\sum_n^N \frac{\sin\left((2n-1)\pi x/L\right)}{(2n-1)^3} \qquad \text{for } n = 1, 2, 3, \cdots$$

where the indexing properly treats only the nonzero coefficients.

As before, the approximation given in the last expression has been written as a finite expansion, where N represents the number of nonzero terms used. This finite expansion is evaluated in the Matlab file **eigenf2.m**, which is listed in Table 9.2. The computational algorithm used here is a little more efficient than the one used for Case 1 (see **eigenf1.m** in Table 9.1 for comparison). Each new term is added to the previous partial sum and a check is made to determine if the additional term has a non-negligible effect on the running sum. If it does, a new term is added and the process is continued up to some maximum number of terms. If the relative contribution of the last term in the partial sum falls below some user set tolerance, the summing loop is stopped, and the final converged Fourier expansion is plotted against the exact function, f(x) (for comparison purposes).

For the Case 2 quadratic relationship, $f(x) = x(L-x)$, 17 terms were needed for convergence to within a relative tolerance of 0.001, and the final converged expansion function after 17 terms is plotted in Fig. 9.2. Note that, this time, the Fourier expansion agrees exactly with the desired f(x). This was expected because the function endpoints and the eigenfunction endpoint are identical and, with enough terms in the convergent series, we expect to get an exact match.

In summary, the two examples given here show the two most common situations that occur with Generalized Fourier Series expansions. If the boundary points of the desired function match the eigenfunctions, then convergence over the whole domain is expected. However, if the endpoint values do not match, complete convergence can never be achieved. Also, concerning an algorithm for evaluating the series, the one illustrated in Table 9.2 for the Case 2 example is clearly the better method -- since it only uses as many terms as necessary for the desired level of accuracy.

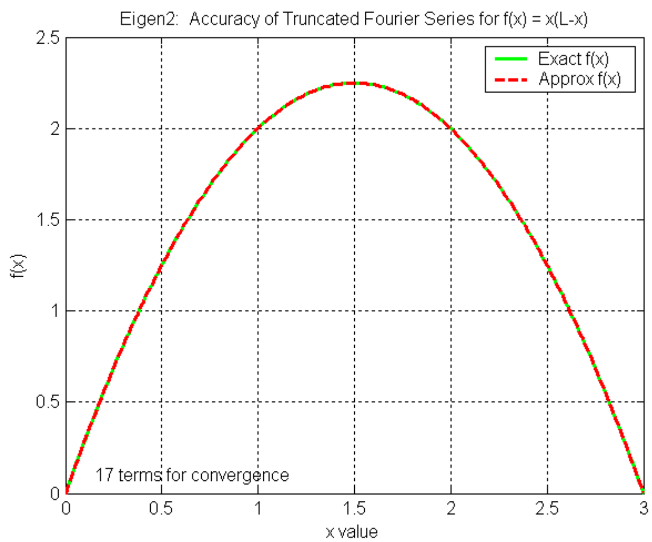

Fig. 9.2 The converged Fourier series for quadratic f(x).

Table 9.2 Listing of Matlab file eigenf2.m.

```
%
%   EIGENF2.M   Demo of Fourier Series Representation for f(x) = x(L-x)
%
%   The goal here is to evaluate the infinite series expansion for
%   f(x) = x(L-x) in terms of sinusoids (i.e. a Fourier series expansion)
%
%   For this problem, the coefficients and basis functions are (see notes):
%      an = (8*L^2/pi^3)/m^3           gn(x) = sin(m*pi*x/L)
%   with m = (2n-1)   and   n = 1, 2, ...
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all, close all
%
%   set x domain and some constants
      L = 3;   Nx = 201;  x = linspace(0,L,Nx);   c = 8*L^2/pi^3;
%
%   calc exact function
      fe = x.*(L - x);
%
%   calc Fourier Series approx
      fa = zeros(size(x));
      Max = 100;   tol = 0.001;   mrerr = 1.0;   n = 0;
      while mrerr > tol   &    n < Max
        n = n+1;  m = 2*n-1;
        ff = (c/m^3)*sin(m*pi*x/L);   fa = fa + ff;
        i = find(fa);                  % finds indices of nonzero values of fa
        mrerr = max(abs(ff(i)./fa(i)));  % compute max relative error
      end
%
%   plot curves
      plot(x,fe,'g-',x,fa,'r--','LineWidth',2),grid
      title('Eigen2:  Accuracy of Truncated Fourier Series for f(x) = x(L-x)')
      ylabel('f(x)'), xlabel('x value')
      legend('Exact f(x)','Approx f(x)')
      if n == Max,  gtext('not converged');  end
      if n < Max,   gtext([num2str(n),' terms for convergence']);  end
%
%   end of demo
```

Example 9.2 -- Generalized Fourier Series Solution to BVPs

Problem Description:

Solve the following BVPs using a Fourier series and compare the resultant infinite series solutions to the exact solutions obtained via the methods of Section II.

Case 1: $y'' + 4y = 1$ with $y(0) = 0$ and $y(L) = 0$

Case 2 $y'' + 4y = x(L-x)$ with $y(0) = 0$ and $y(L) = 0$

Problem Solution:

Method I: Simple Solution as Linear Constant Coefficient ODEs

First we note the each of these problems involve a simple linear ODE with constant coefficients. Thus, the methods reviewed in Section II are applicable and we can easily find the homogeneous, particular, and general solutions to each case. Then, by applying the two explicit Dirichlet boundary conditions, we can obtain the unique solution to each problem. The detailed steps involved in this process are given below:

Case 1

The characteristic equation for an assumed homogeneous solution of the form $y_h = e^{rx}$ is $r^2 + 4 = 0$, with imaginary roots $r_{1,2} = \pm 2i$. This gives the homogeneous solution

$$y_h(x) = A_1 \cos 2x + A_2 \sin 2x$$

Now, for a particular solution, we assume $y_p(x) = A_3$ and, upon substitution into the original ODE, we have

$$4A_3 = 1 \quad \text{or} \quad A_3 = \frac{1}{4}$$

Thus, combining y_h and y_p gives the general solution

$$y_1(x) = A_1 \cos 2x + A_2 \sin 2x + \frac{1}{4}$$

Now, applying the BCs, gives

$$y(0) = A_1 + \frac{1}{4} = 0 \quad \text{or} \quad A_1 = -\frac{1}{4}$$

$$y(L) = A_1 \cos 2L + A_2 \sin 2L + \frac{1}{4} = 0$$

or $$A_2 \sin 2L = -\frac{1}{4} + \frac{1}{4}\cos 2L = \frac{1}{4}\left(\cos 2L - 1\right)$$

and, solving explicitly for A_2 gives,

$$A_2 = \frac{\cos 2L - 1}{4\sin 2L}$$

With a specified value of L, A_1 and A_2 are simple numbers, and we get an exact (i.e. unique) solution to the Case 1 BVP:

$$y_1(x) = -\frac{1}{4}\cos 2x + \frac{\cos 2L - 1}{4\sin 2L}\sin 2x + \frac{1}{4}$$

Case 2

This problem, of course, has the same homogeneous solution as Case 1. The particular solution is different, however, and it can be developed as follows. Let $y_p(x)$ be a simple polynomial of the same form as the forcing function and all it derivatives, or

$$y_p(x) = A_3x^2 + A_4x + A_5$$

$$y_p'(x) = 2A_3x + A_4$$

$$y_p''(x) = 2A_3$$

Then, upon substitution into the defining ODE, we have

$$2A_3 + 4\left(A_3x^2 + A_4x + A_5\right) = Lx - x^2$$

This gives three simple equations with the following solution:

$$A_3 = -\frac{1}{4}, \qquad A_4 = \frac{L}{4}, \qquad \text{and} \qquad A_5 = -\frac{2A_3}{4} = \frac{1}{8}$$

Thus, the general solution to the Case 2 problem becomes

$$y_2(x) = A_1\cos 2x + A_2\sin 2x + \frac{1}{8} + \frac{L}{4}x - \frac{1}{4}x^2$$

Now, applying the BCs for this problem gives

$$y(0) = A_1 + \frac{1}{8} = 0 \qquad \text{or} \qquad A_1 = -\frac{1}{8}$$

$$y(L) = A_1\cos 2L + A_2\sin 2L + \frac{1}{8} + \frac{L^2}{4} - \frac{L^2}{4} = 0$$

which gives

$$A_2 = \frac{\frac{1}{8}\left(\cos 2L - 1\right)}{\sin 2L} = \frac{\cos 2L - 1}{8\sin 2L}$$

And, upon evaluation and substitution into the general solution, we have the exact solution to the Case2 BVP:

$$y_2(x) = -\frac{1}{8}\cos 2x + \frac{\cos 2L - 1}{8\sin 2L}\sin 2x + \frac{1}{8} + \frac{L}{4}x - \frac{1}{4}x^2$$

Method II: Solution using Generalized Fourier Series

Recall that Example 9.1 involved an eigenvalue problem of the form

$$y'' + \lambda y = 0 \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y(L) = 0$$

and that the normalized eigenfunctions were given by

$$y_n(x) = \sqrt{\frac{2}{L}} \sin n\pi x/L$$

Since the BCs for the BVPs of interest here are identical to those from Example 9.1, it makes sense to expect that a solution of the form

$$y(x) = \sum_n b_n y_n(x) = \sum_n b_n \sqrt{\frac{2}{L}} \sin n\pi x/L$$

may be appropriate.

Now, to actually solve the ODE, we make this assumption and expand the RHS forcing function in terms of a Fourier series. For the two forcing functions of interest here, we have the following relationships (see Example 9.1):

$$1 = \frac{4}{\pi} \sum_n \frac{\sin n\pi x/L}{n} \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

$$x(L-x) = \frac{8L^2}{\pi^3} \sum_n \frac{\sin n\pi x/L}{n^3} \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

Now, substituting these relationships into the ODE for each case should allow us to determine the b_n specific to each case of interest here. Let's do this explicitly for each problem, where we note that, by differentiation term by term, we have

$$y(x) = \sum_n b_n \sqrt{\frac{2}{L}} \sin n\pi x/L$$

$$y'(x) = \sum_n b_n \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L}\right) \cos n\pi x/L$$

and

$$y''(x) = -\sum_n b_n \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L}\right)^2 \sin n\pi x/L$$

Case 1

Now, making explicit substitutions into $y'' + 4y = 1$, we have

$$-\sum_n b_n \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L}\right)^2 \sin n\pi x/L + 4\sum_n b_n \sqrt{\frac{2}{L}} \sin n\pi x/L = \frac{4}{\pi} \sum_n \frac{1}{n} \sin n\pi x/L \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

And, equating the coefficients of like terms gives

$$\left(4\sqrt{\frac{2}{L}}-\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)^2\right)b_n=\frac{4}{\pi}\frac{1}{n}=\frac{2L}{\pi}\sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\frac{1}{n}$$

or

$$b_n=\frac{2L}{\pi}\sqrt{\frac{2}{L}}\frac{1/n}{4-(n\pi/L)^2}\qquad \text{for}\quad n=1,3,5,\cdots$$

where we note that, since the RHS is zero for even n, the b_n are also zero for even n.

Thus, we can write the Fourier series solution for the Case 1 BVP as

$$y_1(x)=\sum_n\frac{2L}{\pi}\sqrt{\frac{2}{L}}\frac{1/n}{4-(n\pi/L)^2}\sqrt{\frac{2}{L}}\sin n\pi x/L\qquad \text{for}\quad n=1,3,5,\cdots$$

or

$$y_1(x)=\frac{4}{\pi}\sum_n\frac{1/n}{4-(n\pi/L)^2}\sin n\pi x/L\qquad \text{for}\quad n=1,3,5,\cdots$$

where, as before, we will replace n with 2n-1 during actual implementation to allow the use of all the positive integers, $n=1,2,3,\cdots$.

Case 2

Now making explicit substitutions into $y''+4y=x(L-x)$, we have

$$-\sum_n b_n\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)^2\sin n\pi x/L+4\sum_n b_n\sqrt{\frac{2}{L}}\sin n\pi x/L=\frac{8L^2}{\pi^3}\sum_n\frac{1}{n^3}\sin n\pi x/L\quad \text{for}\ \ n=1,3,5,\cdots$$

which gives

$$\left(4\sqrt{\frac{2}{L}}-\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)^2\right)b_n=\frac{8L^2}{\pi^3}\frac{1}{n^3}=\frac{4L^3}{\pi^3}\sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\frac{1}{n^3}$$

or

$$b_n=\frac{\frac{4L^3}{\pi^3}\sqrt{\frac{2}{L}}\frac{1}{n^3}}{4-(n\pi/L)^2}\qquad \text{for}\quad n=1,3,5,\cdots$$

where again, since the RHS is zero for even n, the b_n are also zero for even n.

Thus, we can write the Fourier series solution for the Case 2 BVP as

$$y_2(x)=\sum_n\frac{\frac{4L^3}{\pi^3}\sqrt{\frac{2}{L}}\frac{1}{n^3}}{4-(n\pi/L)^2}\sqrt{\frac{2}{L}}\sin n\pi/L\qquad \text{for}\quad n=1,3,5,\cdots$$

or

$$y_2(x) = \frac{8L^2}{\pi^3}\sum_n \frac{1/n^3}{4-(n\pi/L)^2}\sin n\pi/L \qquad \text{for} \quad n = 1, 3, 5, \cdots$$

Evaluation and Comparison of the Methods

The resultant mathematical expressions for $y_1(x)$ and $y_2(x)$ for the two methods used here are quite different. A relatively simple analytical expression in terms of sines, cosines, and polynomial functions resulted from the traditional analytical solution method for constant coefficient ODEs. In contrast, the generalized Fourier series approach gives a solution in the form of an infinite series where, in this case, a simple sine function was used as the basis functions in the eigenfunction expansion. However, it is not at all obvious that the solutions for the two methods are equivalent, and the actual shapes of the Case 1 and Case 2 solutions, $y_1(x)$ and $y_2(x)$, are not readily apparent -- especially for the Fourier series solution methodology.

To show that the methods give identical results and to observe the actual functional behavior over the interval of interest, the above solutions were implemented, evaluated, and plotted using Matlab. In particular, the Case 1 solutions are compared in m-file **eigenf3.m** and the Case 2 results are evaluated in script file **eigenf4.m**. These files are listed in Tables 9.3 and 9.4 and the summary plots for Case 1 and Case 2 are shown in Figs. 9.3 and 9.4, respectively.

The actual Matlab programs are quite straightforward, with the evaluation of the Fourier series following the same algorithm as implemented previously in Example 9.1 (see the **eigenf2.m** file in Table 9.2). The first observation to make from the plots is that the truncated Fourier series solution method gives an approximate solution that is essentially exact for both cases (i.e. the two curves in Figs. 9.3 and 9.4 overlap). This, however, is consistent with expectations since, in both cases, the boundary values for $y_1(x)$ and $y_2(x)$ correspond exactly with the chosen basis functions within the generalized Fourier series. Thus, as we saw in Example 9.1, if the boundary values from the solution to the BVP match the endpoint values for the eigenfunctions, then convergence over the whole domain is expected.

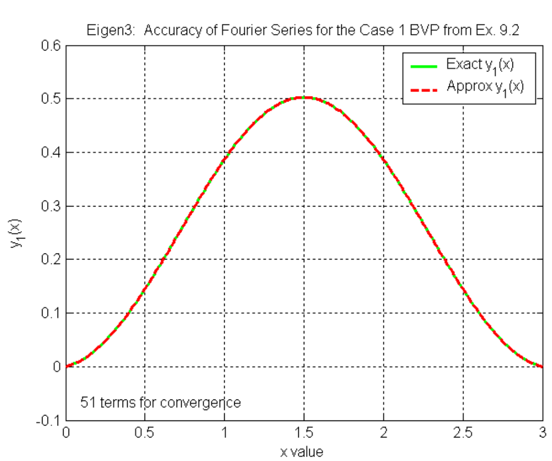

Fig. 9.3 Solutions for the Case 1 BVP.

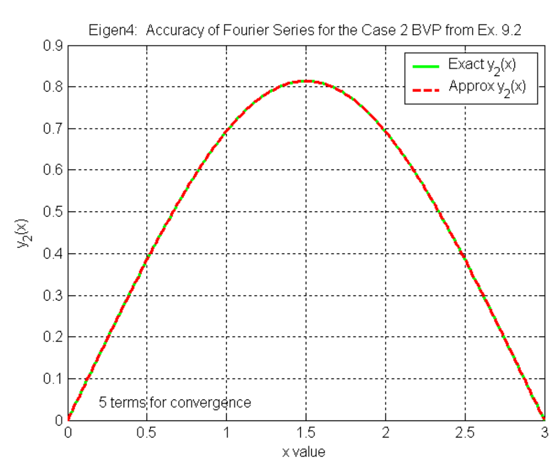

Fig. 9.4 Solutions for the Case 2 BVP.

Note, however, that even though convergence was achieved in both cases, the Case 1 solution required significantly more terms in the expansion (51 terms for the Case 1 BVP and only 5 terms for the Case 2 solution). This behavior is associated with the eigenfunction representation of the RHS forcing function within the ODE of interest. In Example 9.1, we have seen that the Fourier series representation of $f_2(x) = x(L - x)$ is essentially exact, but that the expansion of $f_1(x) = 1$ can never converge over the whole domain. Thus, when solving the inhomogeneous BVP, the nature of the RHS source can certainly affect the rate of convergence of the Fourier series for the solution -- that is, if it takes many terms to get a good representation of the RHS forcing function, then the series solution for y(x) may also need many terms. Note, however, that completeness of the RHS function, f(x), is not a requirement for completeness of the Fourier series for the solution, y(x), to the BVP -- it is the consistency of the BCs on y(x) and the endpoint conditions associated with the eigenfunctions that is important here. Thus, in this example, since the chosen eigenfunctions satisfy the BCs for both BVPs, convergence to the exact solution was indeed expected for both cases. The convergence behavior of the Fourier series for the forcing function, f(x), can certainly affect the observed rate of convergence of y(x), but it is the actual BCs that influence whether or not complete convergence should be expected.

Table 9.3 Listing of Matlab file eignf3.m (Example 9.2 Case 1).

```
%
%   EIGENF3.M   Demo for using Fourier Series for solving BVPs
%
%   The goal here is to use Fourier Series to solve the simple BVP given by
%        y'' + 4y = 1   with   y(0) = 0   and   y(L) = 0
%   and to compare the approximate series solution with the exact solution (see
%   Case 1 in Example 9.2 in the Math Method Notes for details).
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all, close all
%
%   set x domain
      L = 3;   Nx = 201;  x = linspace(0,L,Nx);
%
%   calc exact solution
      ce = (cos(2*L)-1)/(4*sin(2*L));
      ye = -(1/4)*cos(2*x) + ce*sin(2*x) + 1/4;
%
%   calc Fourier Series approx to y(x)
      c = 4/pi;   ya = zeros(size(x));
      Max = 100;   tol = 0.001;   mrerr = 1.0;   n = 0;
      while mrerr > tol   &    n < Max
        n = n+1;  m = 2*n-1;
        ff = (c/m)/(4-(m*pi/L)^2)*sin(m*pi*x/L);   ya = ya + ff;
        i = find(ya);                    % finds indices of nonzero values of fa
        mrerr = max(abs(ff(i)./ya(i)));  % compute max relative error
      end
%
%   plot curves
      plot(x,ye,'g-',x,ya,'r--','LineWidth',2),grid
      title('Eigen3:  Accuracy of Fourier Series for the Case 1 BVP from Ex. 9.2')
      ylabel('y_1(x)'), xlabel('x value')
      legend('Exact y_1(x)','Approx y_1(x)')
      if n == Max,  gtext('not converged');  end
      if n < Max,   gtext([num2str(n),' terms for convergence']);  end
%
%   end of demo
```

Table 9.4 Listing of Matlab file eigenf4.m (Example 9.2 Case 2).

```
%
%   EIGENF4.M   Demo for using Fourier Series for solving BVPs
%
%   The goal here is to use Fourier Series to solve the simple BVP given by
%        y'' + 4y = x(L-x)   with   y(0) = 0   and   y(L) = 0
%   and to compare the approximate series solution with the exact solution (see
%   Case 2 in Example 9.2 in the Math Method Notes for details).
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all, close all
%
%   set x domain
      L = 3;   Nx = 201;  x = linspace(0,L,Nx);
%
%   calc exact solution
      ce = (cos(2*L)-1)/(8*sin(2*L));
      ye = -(1/8)*cos(2*x) + ce*sin(2*x) + 1/8 + (L/4)*x -(1/4)*x.*x;
%
%   calc Fourier Series approx to y(x)
      c = 8*L^2/pi^3;   ya = zeros(size(x));
      Max = 100;   tol = 0.001;   mrerr = 1.0;   n = 0;
      while mrerr > tol   &    n < Max
        n = n+1;  m = 2*n-1;
        ff = (c/m^3)/(4-(m*pi/L)^2)*sin(m*pi*x/L);   ya = ya + ff;
        i = find(ya);                    % finds indices of nonzero values of fa
        mrerr = max(abs(ff(i)./ya(i)));  % compute max relative error
      end
%
%   plot curves
      plot(x,ye,'g-',x,ya,'r--','LineWidth',2),grid
      title('Eigen4:  Accuracy of Fourier Series for the Case 2 BVP from Ex. 9.2')
      ylabel('y_2(x)'), xlabel('x value')
      legend('Exact y_2(x)','Approx y_2(x)')
      if n == Max,  gtext('not converged');  end
      if n < Max,   gtext([num2str(n),' terms for convergence']);  end
%
%   end of demo
```

Orthogonality of the Eigenfunctions

The General Case

We have stated that the eigenfunctions of the Sturm-Liouville problem are orthogonal and, in Example 9.1, we have shown this property for a particular situation. However, since orthogonality is such an important property, it is important to examine its development for the general Sturm-Liouville problem. Thus, the present subsection treats the general case and much of the remainder of this section shows how Example 9.1, the Legendre polynomials, and the ordinary Bessel functions fit into the general development. These three specific illustrations cover the most common situations that occur in practical applications.

To start the proof of orthogonality for the general case, we simply rewrite the general Sturm-Liouville problem [see eqns. (9.1) and (9.2)] using a shorthand notation, or

$$[ry']' + [q + \lambda p]y = 0$$

with boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0$$
$$k_1 y(b) + k_2 y'(b) = 0$$

We now let $y_m(x)$ and $y_n(x)$ be eigenfunctions for two different eigenvalues, λ_m and λ_n (i.e. $m \neq n$). Then the defining equations for $y_m(x)$ and $y_n(x)$ are:

$$[ry_m']' + [q + \lambda_m p]y_m = 0 \tag{9.3}$$

$$[ry_n']' + [q + \lambda_n p]y_n = 0 \tag{9.4}$$

Multiplying eqn. (9.3) by y_n and eqn. (9.4) by y_m and subtracting the resultant expressions give

$$y_n[ry_m']' - y_m[ry_n']' + (\lambda_m - \lambda_n)py_m y_n = 0$$

or

$$(\lambda_m - \lambda_n)py_m y_n = y_m[ry_n']' - y_n[ry_m']' \tag{9.5}$$

The right hand side (RHS) of eqn. (9.5) can be written as

$$\text{RHS of eqn. (9.5)} = [(ry_n')y_m - (ry_m')y_n]' \tag{9.6}$$

One can show this latter relationship by performing the indicated operations, or

$$[(ry_n')y_m - (ry_m')y_n]' = (ry_n')y_m' + [ry_n']'y_m - (ry_m')y_n' - [ry_m']'y_n$$

and, since the first and third terms cancel, we are left with the expression in eqn. (9.6).

Now, combining eqns. (9.5) and (9.6) and integrating over the interval $a \le x \le b$ gives

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b py_m y_n dx &= \left[(ry_n')y_m - (ry_m')y_n\right]\Big|_a^b \\ &= r(b)\left[y_n'(b)y_m(b) - y_m'(b)y_n(b)\right] \\ &\quad - r(a)\left[y_n'(a)y_m(a) - y_m'(a)y_n(a)\right]\end{aligned} \tag{9.7}$$

Since $\lambda_m \neq \lambda_n$, for orthogonality, the right hand side of eqn. (9.7) must vanish. Thus, we see that the boundary conditions usually play an important role in establishing orthogonality (as well as in defining the eigenfunctions originally). Note that many different combinations for the BCs at the two boundary points, a and b, will force the RHS of eqn. (9.7) to zero. However, for the special case of r(a) and r(b) both zero, the boundary conditions play no role in showing orthogonality -- and, for this case, the solutions of the Sturm-Liouville problem are orthogonal independent of the boundary conditions imposed on the problem. Also note that, in general, orthogonality is with respect to the weight function p(x). These observations are very important -- and the student should have a good understanding of the development and use of the general orthogonality conditions implied in eqn. (9.7). The following three cases illustrate its use to establish orthogonality for three different situations.

Example 9.1 Revisited

In Example 9.1, the ODE of interest was

$$y'' + \lambda y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad y(L) = 0$$

This is a specific case of a general Sturm-Liouville problem with

$$r(x) = 1 \quad q(x) = 0 \quad p(x) = 1 \quad \text{and} \quad a = 0 \quad b = L$$

Using eqn. (9.7) to establish orthogonality gives

$$\int_0^L y_m(x) y_n(x) dx = (1)\left[y_n'(L) y_m(L) - y_m'(L) y_n(L)\right] - (1)\left[y_n'(0) y_m(0) - y_m'(0) y_n(0)\right]$$

and since the boundary conditions are y(0) = 0 and $y(L) = 0$, every term on the right hand side vanishes identically. Therefore, $\int_0^L y_m y_n dx = 0$, as shown previously for Example 9.1.

Legendre Polynomials

Recall that Legendre’s equation was written as

$$(1 - x^2) y_n'' - 2x y_n' + n(n+1) y_n = 0$$

but this is equivalent to

$$[(1 - x^2) y_n']' + \lambda_n y_n = 0 \qquad \text{where} \qquad \lambda_n = n(n+1)$$

Therefore, this is a Sturm-Liouville problem with

$$r(x) = (1 - x^2) \quad q(x) = 0 \quad \text{and} \quad p(x) = 1$$

Also we note that the range of interest here is $a \le x \le b$ with a = -1 and b = 1 and that

$$r(a) = r(-1) = 0 \qquad \text{and} \qquad r(b) = r(1) = 0$$

Equation (9.7) shows that the solutions to Legendre’s equation are indeed orthogonal and that no specific boundary conditions are needed to force orthogonality [since r(a) and r(b) are already both zero].

Ordinary Bessel Functions

The ordinary Bessel equation is given as

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$$

but this is equivalent to (dividing by x)

$$[xy']' + \left(\lambda x + \frac{-\nu^2}{x}\right)y = 0 \qquad \text{where} \quad \lambda = \alpha^2$$

Therefore, this is a Sturm-Liouville problem with

$$r(x) = x \qquad q(x) = \frac{-\nu^2}{x} \qquad \text{and} \qquad p(x) = x$$

Thus, orthogonality will be with respect to the weight function p(x) = x and the boundary conditions must be such that the right hand side of eqn. (9.7) vanishes.

To elaborate a little, let's limit our discussion to integer order Bessel functions, or let $\nu \to n$ (this is the usual case). Then the eigenvalues, $\lambda_{mn} = \alpha_{mn}^2$, represent the infinite number of values of λ_{mn} for $m = 1, 2, \cdots$ for the ordinary Bessel functions of order n that satisfies the specific boundary conditions for a given problem.

Writing $y(x) = J_n(\alpha_{mn} x)$, we see that eqn. (9.7), for this case, becomes

$$\begin{aligned}(\alpha_{mn}^2 - \alpha_{kn}^2)\int_a^b xJ_n(\alpha_{mn}x)J_n(\alpha_{kn}x)dx = \\ b\left[J_n'(\alpha_{kn}b)J_n(\alpha_{mn}b) - J_n'(\alpha_{mn}b)J_n(\alpha_{kn}b)\right] \\ -a\left[J_n'(\alpha_{kn}a)J_n(\alpha_{mn}a) - J_n'(\alpha_{mn}a)J_n(\alpha_{kn}a)\right]\end{aligned}$$

Clearly, since both boundary points, a and b, cannot be zero, orthogonality can only be specified by appropriate boundary conditions on the problem. As shown below in Example 9.3, the specific values of $\lambda_{mn} = \alpha_{mn}^2$ that satisfy these conditions are the eigenvalues of the problem and they are related to the zeros of the $J_n(x)$ function -- that is, α_{mn} is interpreted as the m^{th} value of x such that $J_n(x) = 0$. The reader should see the previous discussion on Bessel functions in Section VIII for more information [for example, one should recall that $J_n(x)$ has an infinite number of zeros, etc,]. Also, one should study Example 9.3 in detail as a specific application that illustrates orthogonality for the ordinary Bessel functions.

Example 9.3 -- Neutron Diffusion in a Nuclear Reactor

Problem Description:

Determine the 1-group neutron flux distribution in an infinitely long homogeneous bare critical cylindrical reactor. Also show that the resultant eigenfunctions for this problem are orthogonal and find the norm of these orthogonal eigenfunctions.

Problem Solution:

Background

This example is from basic Nuclear Reactor Theory. In a nuclear reactor, ^{235}U fissions by neutron bombardment and recoverable energy to drive a steam power plant is released. Neutrons are also released in the fission process, and these are used to initiate additional fission reactions. For steady state operation, the parasitic neutron losses in the system and the neutron absorption in the uranium fuel are balanced exactly by the neutron production rate from fission. Since it is the neutron distribution that determines the various interaction rates and the energy production rate, it becomes important when designing and operating a nuclear reactor to determine the neutron population throughout the system. One must solve a second order differential balance equation to obtain the desired neutron density or flux distribution. The simplest form of the neutron balance equation is the so-called ***Neutron Diffusion Equation***.

One of the steady state critical ideal reactor geometries that can be treated via analytical means is a long bare cylindrical core model, as illustrated in Fig. 9.5. All the adjectives used to describe the system are needed to reduce the general, very complicated, particle balance equation into a form that can be treated analytically. For example, the phrase, ***steady state critical***, indicates that the neutron population is constant in time and that the reaction is self-sustaining (i.e. the neutrons given off in the fission process are balanced perfectly so that they cause the same number of fissions in each generation, which produces the same number of new neutrons, and keeps the total neutron population constant -- on the average). The term ***bare reactor*** means that the fueled core region is surrounded by a vacuum, giving an outer boundary condition of zero flux (the ***neutron flux*** is a measure of the neutron population). The ***long homogeneous*** description implies that the axial height is large relative to the radius and that the material properties are constant throughout the system. These conditions suggest that the variation of the neutron density in the axial and azimuthal directions is negligible, leaving a functional dependence on only one variable, or $\phi(r,\theta,z) \Rightarrow \phi(r)$, where ϕ is the symbol used to represent the neutron flux. Also, symmetry in the system suggests that the neutron population will be the largest in the center of the reactor, which implies that the flux gradient is zero at r = 0.

The Neutron Flux Distribution

With all the above conditions and simplifications, the Neutron Diffusion Equation for a one energy group (assumes all the neutrons have the same average energy) bare critical reactor model becomes

$$\nabla^2\phi + B^2\phi = 0$$

where the ∇^2 or Laplacian operator for 1-D cylindrical geometry is given by

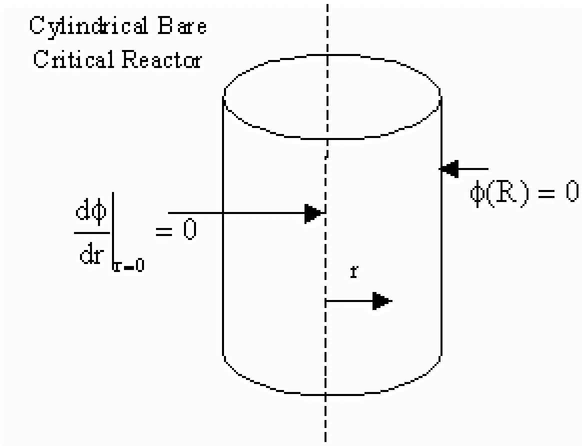

Fig. 9.5 Basic geometry for the cylindrical bare critical reactor in Example 9.3.

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

and B^2 is a constant that is related to material properties of the homogenous system.

Therefore, the particle balance equation for the 1-g 1-D neutron flux distribution is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + B^2\phi = 0$$

with boundary conditions as shown in Fig. 9.5, or

$$\text{at } r = 0,\ \left.\frac{d\phi}{dr}\right|_{r=0} = 0 \qquad \text{and} \qquad \text{at } r = R,\ \phi(R) = 0$$

In mathematical terms, this system is a 2^{nd} order homogeneous ODE with homogeneous boundary conditions. From our recent discussions, we recognize this as an eigenvalue or Sturm-Liouville problem!

To avoid any confusion with the above notation, let's first convert this system into a standard Sturm-Liouville problem, using the notation from previous sections. In particular, if we let $r \Rightarrow x$, $\phi \Rightarrow y$, and $\lambda = B^2$, then, after multiplication by r, the balance equation becomes

$$[xy']' + \lambda xy = 0$$

This is clearly a Sturm-Liouville problem with

$$r(x) = x \qquad q(x) = 0 \qquad \text{and} \qquad p(x) = x$$

and boundary conditions

$$y'(0) = 0 \qquad \text{and} \qquad y(R) = 0$$

We should also recognize this as an ordinary Bessel equation with $\nu = 0$. This can be seen more easily by rewriting the balance equation as

$$x^2y'' + xy' + (\lambda x^2 - 0)y = 0$$

Therefore, the general solution for this problem can be written as

$$y(x) = A_1 J_0\left(\sqrt{\lambda} x\right) + A_2 Y_0(\sqrt{\lambda} x)$$

or

$$y(x) = A_1 J_0(Bx) + A_2 Y_0(Bx)$$

Also, we can compute the gradient, $y'(x)$, as

$$y'(x) = -A_1 B J_1(Bx) - A_2 B Y_1(Bx)$$

where the standard derivative formulas for the ordinary Bessel functions have been used, or

$$J_0'(Bx) = -BJ_1(Bx) \qquad \text{and} \qquad Y_0'(Bx) = -BY_1(Bx)$$

As usual, we now apply the boundary conditions to the general solution. At r = x = 0, we have

$$y'(0) = 0 = -A_1 B \times (0) - A_2 B \times (-\infty)$$

where we have used the facts that $J_1(x)$ approaches zero as $x \to 0$ and $Y_1(x)$ approaches negative infinity as $x \to 0$. Thus, the only way to satisfy this condition is to let $A_2 = 0$. Thus, after applying only the symmetry condition, the solution profile reduces to

$$y(x) = A_1 J_0(Bx)$$

Now, applying the boundary condition at x = R gives

$$y(R) = 0 = A_1 J_0(BR)$$

Since $A_1 = 0$ would give a trivial solution, we must let $J_0(BR) = 0$ to satisfy this condition. However, $J_n(x)$ is an oscillatory function and it has an infinite number of zeros, which we will denote as α_{mn} for $m = 1, 2, \cdots$ -- that is, α_{mn} represents values of x for which $J_n(x) = 0$.

Therefore, from the above discussion, we have

$$J_0(BR) = J_0(\alpha_{m0}) = 0$$

as the ***eigencondition*** for this problem. This leads to

$$B_m = \frac{\alpha_{m0}}{R}$$

as the allowed eigenvalues (really $\lambda_m = B_m^2$) with $m = 1, 2, \cdots$.

With the eigenvalues known [these can be approximated from a plot of $J_0(x)$ or, more accurately, from a root finding algorithm applied to $J_0(x)$], the eigenfunctions are simply

$$y_m(x) = \phi_m(x) = J_0(B_m x) = J_0\left(\frac{\alpha_{m0}}{R} x\right)$$

Note that, for the real physical system, the neutron flux must be real and non-negative, and the only eigenfunction that is positive over the full domain, $0 \le r \le R$, is related to the fundamental mode (i.e. first eigenfunction). Since the first zero of the $J_0(x)$ Bessel function occurs at x = 2.4048, the real neutron flux distribution in the physical system is

$$\phi(r) = A_1 J_0\left(\frac{2.405}{R} r\right)$$

where A_1 is a normalization that is usually determined from the overall power level of the reactor. This latter expression is the desired solution to the first part of the problem description for Example 9.3 (see above).

Orthogonality of the Eigenfunctions

In addition to finding the physical neutron flux profile, from a mathematical viewpoint, it would also be nice to demonstrate that the eigenfunction solutions are indeed orthogonal functions with respect to the weight function p(x) = x. If we take the general orthogonality relationship given in eqn. (9.7) for the general Sturm-Liouville problem and apply it to this problem, we have

$$\int_0^R x y_m y_n dx = \int_0^R x J_0\left(\frac{\alpha_{m0}}{R} x\right) J_0\left(\frac{\alpha_{n0}}{R} x\right) dx$$

$$= R\left[J_0'(\alpha_{n0}) J_0(\alpha_{m0}) - J_0'(\alpha_{m0}) J_0(\alpha_{n0})\right] - 0\left[J_0'(0)J_0(0) - J_0'(0)J_0(0)\right]$$

The first term on the RHS of this expression vanishes because, by definition, $J_0(\alpha_{m0}) = 0$ for $m = 1, 2, \cdots$, and the second term is zero for two reasons -- because of the zero coefficient and because $J_0'(0) = -BJ_1(0) = 0$. Therefore, since the full RHS vanishes, the ordinary Bessel functions are indeed orthogonal with respect to p(x) = x, or

$$\int_0^R x J_0\left(\frac{\alpha_{m0}}{R} x\right) J_0\left(\frac{\alpha_{n0}}{R} x\right) dx = 0 \qquad \text{for} \quad m \ne n$$

Finally, let's compute the normalization required when $m = n$. In this case we need to evaluate the following integral:

$$\|y_m\|^2 = \int_0^R x J_0^2\left(\frac{\alpha_{m0}}{R} x\right) dx = ?$$

To accomplish this we first obtain from the literature the following integral relationship involving the square of the J_0 Bessel function:

$$\int x^{2k+1} J_k^2(x) dx = \frac{x^{2k+2}}{4k+2}\left[J_k^2(x) + J_{k+1}^2(x)\right]$$

and, for k = 0, this becomes

$$\int x J_0^2(x) dx = \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right]$$

Now to evaluate the norm of the eigenfunction, we need to put the desired integral into this form. Therefore, we let

$$z = \frac{\alpha_{m0}}{R}x \qquad \text{and} \qquad dz = \frac{\alpha_{m0}}{R}dx$$

and substitution into the expression for the normalization gives

$$\begin{aligned}\|y_m\|^2 &= \int_0^R xJ_0^2\left(\frac{\alpha_{m0}}{R}x\right)dx = \int_0^{\alpha_{m0}}\left(\frac{R}{\alpha_{m0}}\right)^2 zJ_0^2(z)dz \\ &= \left(\frac{R}{\alpha_{m0}}\right)^2\left[\frac{z^2}{2}\left(J_0^2(z)+J_1^2(z)\right)\right]\Bigg|_0^{\alpha_{m0}} = \left(\frac{R}{\alpha_{m0}}\right)^2\left[\frac{\alpha_{m0}^2}{2}\left(J_0^2(\alpha_{m0})+J_1^2(\alpha_{m0})\right)\right] \\ &= \frac{R^2}{2}J_1^2(\alpha_{m0})\end{aligned}$$

where the last equality uses the fact that $J_0\left(\alpha_{m0}\right)=0$. Thus, the desired eigenfunction normalization is given by

$$\|y_m\|^2 = \int_0^R xJ_0^2\left(\frac{\alpha_{m0}}{R}x\right)dx = \frac{R^2}{2}J_1^2(\alpha_{m0})$$

This example completes our current study of the general Sturm-Liouville Problem. You will find that these ideas, especially those related to orthogonality and generalized Fourier series, will be very useful in lots of applications. They are also especially relevant for several techniques for solving PDEs. Thus, we will revisit some of these concepts in the next section on PDEs (especially the section that deals with the analytical solution of PDEs using the Separation of Variables method).