Mathematical Methods (10/24.539)

VIII. Special Functions and Orthogonality

Introduction

If a particular differential equation (usually representing a linear variable coefficient system) and its power series solution occur frequently in applications, one gives them a name and introduces special symbols that define them. The properties of the functions are studied and tabulated and this information becomes a resource that can be exploited by the practicing engineer.

We have seen that linear constant coefficient systems have solutions that can be written in terms of elementary functions (sinusoids, exponentials, etc.). These functions are called elementary because they are treated in detail in introductory algebra, trigonometry, and calculus courses and they are used routinely in a variety of engineering applications. In short, since we are very familiar with these functions, they are easy to work with and we refer to them as elementary functions.

In contrast, functions that we are not as familiar with are more difficult to use in applications (at least initially) and sometimes these are referred to as non-elementary functions, special functions, or transcendental functions. We will use the ***special function*** designation to emphasize their special significance in a variety of engineering applications. Also, once we gain a little experience with these special functions, we will no longer be imitated with their use and the non-elementary connotation will no longer be applicable (for example, using Bessel functions is as easy as using sinusoids, once you become comfortable with their use).

The special feature of the so-called ***special functions*** is a property called ***orthogonality***. In this section of notes, we define this property, briefly identify several functions that share this special characteristic, and provide some additional details for two particular cases (for Legendre polynomials and Bessel functions). A generalization is made to include a full class of problems that have orthogonal functions as their solution - known as ***Sturm-Liouville Problems*** - in the next section.

The current section on special functions and the subject of orthogonality is subdivided as follows:

Orthogonal Functions

Two functions are said to be orthogonal if, when multiplied together and integrated over the domain of interest, the integral becomes zero. The property of orthogonality is usually applied to a class of functions that differ by one or more variables (and usually represent the basis solutions to a homogeneous eigenvalue problem with an infinite number of eigenfunction solutions). For example, we can represent a class of sinusoids as

$$\psi_n(x) = \sin nx \qquad \text{for } n = 1, 2, 3, \cdots \tag{8.1}$$

where n is a positive integer. A particular function might be $f(x) = \psi_2(x) = \sin 2x$. For an arbitrary function belonging to this set, we simply refer to the discrete index n, where the n^{th} function is denoted as $\psi_n(x)$, or the m^{th} function as $\psi_m(x)$, etc..

The orthogonality property can be stated mathematically as

$$\langle g_m g_n \rangle = \int_a^b g_m(x)\, g_n(x)\, dx = \|g_m\|^2 \delta_{mn} = \begin{cases} 0 & m \neq n \\ \|g_m\|^2 & m = n \end{cases} \tag{8.2}$$

where

$$\|g_m\| = \sqrt{\langle g_m{}^2 \rangle} = \text{norm of the function} \tag{8.3}$$

and δ_{mn} is the Kronecker delta function that takes on the value of unity if m = n and a value of zero if $m \neq n$. If $\|g_m\| = 1$, then $g_m(x)$ is said to be an orthonormal function.

The orthogonality property is important because functions with this characteristic are often used to expand arbitrary functions with an infinite series expansion in terms of the given basis functions. For example, the function f(x) can be written in terms of a ***Generalized Fourier Series*** (implies completeness), or

$$f(x) = \sum_{n=1}^{\infty} a_n g_n(x) \tag{8.4}$$

where the a_n are the expansion coefficients.

The orthogonality property comes into play when one tries to determine an expression for the a_n coefficients. To see this, we multiply eqn. (8.4) by the m^{th} function, $g_m(x)$, and integrate over the domain of interest. Doing this gives

$$\langle g_m(x) f(x) \rangle = \sum_{n=1}^{\infty} a_n \langle g_m(x) g_n(x) \rangle = \sum_{n=1}^{\infty} a_n \|g_m\|^2 \delta_{mn} = a_m \|g_m\|^2 \tag{8.5}$$

or

$$a_m = \frac{\langle g_m(x) f(x) \rangle}{\|g_m\|^2} \tag{8.6}$$

where the summation symbol is eliminated in the last equality in eqn. (8.5) because orthogonality forces all the terms in the infinite sum to zero except for the single term where n = m. This simplification is essential in many practical applications, and it would not be possible without the orthogonality property [as defined in eqn. (8.2)]. Thus we will see that this is a very important characteristic.

The Generalized Fourier Series given in eqn. (8.4) is an eigenfunction expansion in terms of a complete set of orthogonal basis functions. The choice of the basis functions is usually determined by the domain of interest and the boundary conditions imposed upon f(x). The basis functions are usually obtained from a ***Sturm-Liouville Problem*** which results in a set of orthogonal eigenfunctions (see the next section for further details). The term completeness implies that the Generalized Fourier Series converges as $n \to \infty$. Although of theoretical interest, a rigorous proof of completeness is quite often unnecessary because the series is almost always truncated to low order in practical problems.

Finally, we note that, in many cases, the basis functions may be orthogonal with respect to a weight function, p(x). This means that

$$\langle p(x) g_m(x) g_n(x) \rangle = \|g_m\|^2 \delta_{mn} = \begin{cases} 0 & m \neq n \\ \|g_m\|^2 & m = n \end{cases} \tag{8.7}$$

where the normalization is given by

$$\|g_m\| = \sqrt{\langle p g_m^2 \rangle} \tag{8.8}$$

For this case the basic series expansion relationship is unchanged [i.e. eqn. (8.4) is the same], but the expression for the expansion coefficients is modified accordingly to give

$$a_m = \frac{\langle p(x) g_m(x) f(x) \rangle}{\|g_m\|^2} \tag{8.9}$$

Summary of Several Special Functions

As indicated, there are a number of special functions that occur frequently in many different fields of application. As a sample, a few of the more important functions and some of their properties are tabulated below. Note that orthogonality is a common characteristic for these special functions.

Legendre Polynomials

Differential Equation (n is a non-negative integer)	$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$
Rodrique's Formula	$P_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$
Generating Function	$\dfrac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n$
Recurrence Relation	$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$
Orthogonality	$\int_{-1}^{1}P_m(x)P_n(x)\,dx=\dfrac{2}{2n+1}\delta_{mn}$

Associated Legendre Functions (for m = 0, these reduce to Legendre Polynomials)

Differential Equation (n and m are non negative integers)	$\left(1-x^2\right)y''-2xy'+\left[n\left(n+1\right)-\dfrac{m^2}{1-x^2}\right]y=0$
Rodrique's Formula	$P_n^m(x)=\left(-1\right)^m\left(1-x^2\right)^{\frac{m}{2}}\dfrac{d^m}{dx^m}P_n(x)$
Orthogonality	$\int_{-1}^{1}P_n^m(x)P_\ell^m(x)\,dx=\dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}\delta_{n\ell}$

Hermite Polynomials

Differential Equation (n is a non-negative integer)	$y''-2xy'+2ny=0$
Rodrique's Formula	$H_n(x)=\left(-1\right)^n e^{x^2}\dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$
Generating Function	$e^{2tx-t^2}=\sum_{n=0}^{\infty}\dfrac{H_n(x)}{n!}t^n$
Recurrence Relation	$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$
Orthogonality	$\int_{-\infty}^{\infty}e^{-x^2}H_m(x)H_n(x)\,dx=2^n n!\sqrt{\pi}\,\delta_{mn}$

Laguerre Polynomials

Differential Equation (n is a non-negative integer)	$xy'' + (1-x)y' + ny = 0$
Rodrique's Formula	$L_n(x) = e^x \dfrac{d^n}{dx^n}\left(x^n e^{-x}\right)$
Generating Function	$\dfrac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$
Recurrence Relation	$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$
Orthogonality	$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \delta_{mn}$

Bessel Functions

Ordinary Bessel Equation	$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0$
General Solution (ordinary)	$y(x) = A_0 J_\nu(\lambda x) + A_1 Y_\nu(\lambda x)$
Modified Bessel Equation	$x^2y'' + xy' - \left(\lambda^2x^2 + \nu^2\right)y = 0$
General Solution (modified)	$y(x) = A_0 I_\nu(\lambda x) + A_1 K_\nu(\lambda x)$
Hankel Functions	$H_\nu(x) = J_\nu(x) \pm iY_\nu(x)$

Note: Several recurrence, derivative, and integral relationships for the Bessel functions are given in a subsequent subsection. Additional relationships and some specific examples are also given in later subsections. The orthogonality properties of the ordinary Bessel functions, which are somewhat complicated because of their relationship to the specified boundary conditions for a given problem, are also treated later in Section IX: The Sturm-Liouville Problem and Generalized Fourier Series.

The Gamma Function

Although not really in the same classification as the ***Special Functions*** summarized in the previous subsection, the so-called ***Gamma Function*** is also a very important function that is encountered frequently in application (and we will need it in subsequent developments). The gamma function is an integral relationship that is defined as follows:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{8.10}$$

This integral is convergent for $n > 0$.

Since integrals of this type occur so frequently, it becomes convenient to develop and tabulate several key relationships for future use. In particular, three such expressions associated with the gamma function are given below.

Gamma Function Relationships

For any positive n	$\Gamma(n+1) = n\Gamma(n)$
For a positive integer	$\Gamma(n+1) = n!$
For n = 1/2	$\Gamma(1/2) = \sqrt{\pi}$

The remainder of this subsection formally develops these three relationships and gives a simple application of their use.

Proof that, for any positive n, $\mathbf{\Gamma(n+1) = n\Gamma(n)}$

To see this, we have from eqn. (8.10) that

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

Now integrating by parts, with $\int u\,dv = uv - \int v\,du$, we let

$$u = x^n \qquad dv = e^{-x}dx \qquad \text{then} \qquad du = nx^{n-1}dx \qquad v = -e^{-x}$$

Therefore,

$$\Gamma(n+1) = -x^n e^{-x}\Big|_0^\infty + \int_0^\infty \left(nx^{n-1}\right) e^{-x} dx$$

$$= -(0-0) + n\int_0^\infty x^{n-1} e^{-x} dx = n\Gamma(n)$$

Proof that, for a positive integer, $\mathbf{\Gamma(n+1) = n!}$

If n is a positive integer, then

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3 \times 2 \times 1 = 3!$$

or, in general, $\Gamma(n+1) = n!$ [where $\Gamma(n)$ is sometimes referred to as the generalized factorial function].

***Proof that, for n = 1/2,* $\mathbf{\Gamma(1/2) = \sqrt{\pi}}$**

Setting n = 1/2 in the basic definition gives,

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx$$

Letting $x = u^2$ gives $dx = 2udu$ and putting this result into the integral reduces the original expression to

$$\Gamma(1/2) = \int_0^\infty u^{-1} e^{-u^2} 2udu = 2\int_0^\infty e^{-u^2} du$$

Squaring this result gives

$$\left[\Gamma(1/2)\right]^2 = 4\left[\int_0^\infty e^{-u^2} du\right]\left[\int_0^\infty e^{-v^2} dv\right] = 4\int_0^\infty \int_0^\infty e^{-\left(u^2+v^2\right)} dudv$$

Now switching to polar coordinates with $u = r\cos\theta$ and $v = r\sin\theta$, we have

$$u^2 + v^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \qquad \text{and} \qquad dudv \to rdrd\theta$$

and with the u,v domain limits defining the first quadrant, $0 < u < \infty$ and $0 < v < \infty$, the limits on r and θ become $0 < r < \infty$ and $0 < \theta < \pi/2$.

Therefore, the above expression becomes

$$\left[\Gamma(1/2)\right]^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = 4\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi$$

Thus, we have shown that $\Gamma\left(1/2\right) = \sqrt{\pi}$.

An Example

As a simple example of the use of the gamma function, consider the following integral,

$$I = \int_0^\infty \sqrt{y} e^{-y^3} dy$$

Letting $x = y^3$ and $dx = 3y^2 dy = 3x^{2/3} dy$, this becomes

$$I = \int_0^\infty \left(x^{1/6}\right) e^{-x} \left(\frac{1}{3}x^{-2/3}\right) dx = \frac{1}{3}\int_0^\infty x^{-1/2} e^{-x} dx = \frac{1}{3}\Gamma\left(1/2\right) = \frac{\sqrt{\pi}}{3}$$

Thus, with the use of the gamma function, evaluating this integral is quite straightforward.

Legendre's Equation and Legendre Polynomials (in more detail)

As an illustration of the kind of manipulations necessary to develop and work with the special functions identified previously, let's expand somewhat our discussion of Legendre's equation and Legendre polynomials. The development and manipulations of the other special functions are handled in a similar manner (especially the various polynomial relationships -- Hermite and Laguerre polynomials, for example).

Solution via Power Series

Recall that Legendre's equation is given by

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0 \tag{8.11}$$

Since eqn. (8.11) is analytic around $x_0 = 0$, we can use the standard power series method to determine y(x). For this case, let

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{8.12}$$

and upon substitution of this form and its appropriate derivative relationships into the original equation, we get the recurrence relation

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m \tag{8.13}$$

where a_0 and a_1 are arbitrary constants and $m = 0, 1, 2, \cdots$. Therefore, we can write the solution to Legendre's equation as

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{8.14}$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n+3)n(n+1)}{4!}x^4 - \cdots \tag{8.15}$$

and

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n+4)(n-1)(n+2)}{5!}x^5 - \cdots \tag{8.16}$$

These series converge for $|x| \le 1$.

Standard Form for Legendre Polynomials

Now, in many applications, n in eqn. (8.11) will be a non-negative integer. But, when this is true, the above expressions [i.e. eqns. (8.15) and (8.16)] reduce to polynomials of order n. In particular, ***$y_1(x)$ is a polynomial of order n if n is even***, and ***$y_2(x)$ is a polynomial of order n if n is odd***. These polynomials, multiplied by some constants, are called ***Legendre Polynomials***.

To put the polynomials into ***standard form***, let's solve the above recurrence relation for a_m, giving

$$a_m = -\frac{(m+2)(m+1)}{(n-m)(n+m+1)} a_{m+2} \tag{8.17}$$

where

$$m \le n-2 \quad \begin{cases} n = \text{even} & m = 0,\ 2,\ 4,\ \cdots \le n-2 \\ n = \text{odd} & m = 1,\ 3,\ 5,\ \cdots \le n-2 \end{cases}$$

Now, instead of writing all the non-vanishing coefficients in terms of a_0 or a_1, let's write them in terms of the coefficient of the highest power of x (i.e. a_n). In particular, choosing a_n as

$$a_n = \frac{(2n)!}{2^n (n!)^2} \tag{8.18}$$

gives

$$P_n(x)\big|_{x=1} = 1 \tag{8.19}$$

for all n where the domain of interest is $-1 \le x \le 1$.

To put the desired polynomials into final form, note that using eqn. (8.17) with m = n-2 gives

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = \frac{-n(n-1)(2n)!}{2(2n-1)2^n (n!)^2}$$

and, after some manipulation, this can be written as

$$a_{n-2} = \frac{(2n-2)!}{2^n (n-1)!(n-2)!}$$

Performing similar manipulations (i.e. some more magic) with m = n-4, eqn. (8.17) can also be written as

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}$$

This procedure can be continued to develop a general relationship for a_{n-2m} for $n-2m \ge 0$, or

$$a_{n-2m} = \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} \tag{8.20}$$

and

$$P_n(x) = \sum_{m=0}^{M} a_{n-2m} x^{n-2m} \tag{8.21}$$

where M = n/2 or M = (n-1)/2, which whichever is an integer.

Note: The above steps, although not completely rigorous, show the basic idea for putting the general solution into ***standard form***. The details here are not overly important, but eqns. (8.19) - (8.21) are indeed important, and they give the so-called ***Legendre Polynomials*** in standard form. The particular form given here is somewhat arbitrary, but it is consistent with most of the literature on this subject.

Some Low Order Legendre Polynomials

Putting specific values into eqns. (8.20) and (8.21) gives (recall that $n-2m \geq 0$):

n	**m**	**$P_n(x)$**
0	0	$P_0(x) = 1$
1	0	$P_1(x) = \frac{(1)(2)!}{2(1)(1)(1)} x = x$
2	0, 1	$P_2(x) = \frac{(1)(4)!}{(4)(1)(2)!(2)!} x^2 + \frac{(-1)(2)!}{(4)(1)(1)(1)} x^0 = \frac{1}{2}\left(3x^2 - 1\right)$
3	0, 1	Etc. (but the algebra gets tedious)

Some Important Relationships

Note that ***Rodrique's Formula*** can also be used to generate explicit formulae for the low order Legendre polynomials. In particular, given Rodrique's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[\left(x^2 - 1\right)^n\right] \tag{8.22}$$

we can develop the low order polynomials as follows:

n	**$P_n(x)$**
0	$P_0(x) = 1$
1	$P_1(x) = \frac{1}{2}\frac{d}{dx}\left(x^2 - 1\right) = \frac{1}{2}(2x) = x$
2	$P_2(x) = \frac{1}{(4)(2)}\frac{d^2}{dx^2}\left[\left(x^2 - 1\right)^2\right] = \frac{1}{8}\frac{d}{dx}\left[2\left(x^2-1\right)2x\right]$ $= \frac{1}{2}\frac{d}{dx}\left[x^3 - x\right] = \frac{1}{2}\left(3x^2 - 1\right)$
3	Etc. (but this also becomes rather tedious)

The best way to generate explicit formulae for the Legendre polynomials is to use one of the many ***Recurrence Relations*** that are available (see any good reference book on mathematical functions for a tabulation of these relationships - the well-known ***Handbook of Mathematical Functions*** by Abramowitz and Stegun is one excellent source, for example). These recurrence relationships are particularly useful for computer evaluation of Legendre polynomials and their derivatives. In particular, two such relations that are widely used are:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{8.23}$$

and

$$\left(x^2-1\right)\frac{d}{dx}P_n(x) = n\left[xP_n(x) - P_{n-1}(x)\right] \tag{8.24}$$

To illustrate the use of eqn. (8.23), let's develop an explicit expression for $P_3(x)$. To do this we simply let n = 2 in the recurrence relationship, or

$$3P_3 = 5xP_2 - 2P_1 = (5x)\frac{1}{2}\left(3x^2-1\right) - 2x = \frac{15x^3}{2} - \frac{9x}{2}$$

Thus,

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \tag{8.25}$$

Since $P_n(x)$ is simply a polynomial of order n, we can easily find first or higher-order derivative information. For example, $P_2'(x)$ is given by

$$P_2' = \frac{d}{dx}\left[\frac{1}{2}\left(3x^2-1\right)\right] = 3x \tag{8.26}$$

A recurrence formula, however, is very handy for computer implementation. Using eqn. (8.24), we can generate this same result with

$$\left(x^2-1\right)P_2' = 2\left[\frac{x}{2}\left(3x^2-1\right) - x\right] = 3x^3 - x - 2x = 3x\left(x^2-1\right)$$

or $P_2' = 3x$ as before.

As indicated previously, the most important special feature of the so-called ***Special Functions*** is their ***Orthogonality Property*** (see subsection on ***Orthogonal Functions***). For the Legendre polynomials this relationship is written as

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn} \tag{8.27}$$

where δ_{mn} is the Kronecker delta function.

Let's ***derive*** this orthogonality relationship formally to show the basic procedure that is used for most developments of this type. Since $P_m(x)$ and $P_n(x)$ satisfy Legendre's equation, we have for $m \neq n$

$$\left(1-x^2\right)P_m'' - 2xP_m' + m\left(m+1\right)P_m = 0 \tag{8.28}$$

$$\left(1-x^2\right)P_n'' - 2xP_n' + n\left(n+1\right)P_n = 0 \tag{8.29}$$

Now multiply eqn. (8.28) by $P_n(x)$ and eqn. (8.29) by $P_m(x)$ and subtract the resultant expressions giving

$$\left(1-x^2\right)\left(P_nP_m'' - P_mP_n''\right) - 2x\left(P_nP_m' - P_mP_n'\right) + \left[m(m+1) - n(n+1)\right]P_mP_n = 0$$

but

$$\frac{d}{dx}\left(P_nP_m{}' - P_mP_n{}'\right) = P_nP_m{}'' + P_n{}'P_m{}' - P_mP_n{}'' - P_m{}'P_n{}' = P_nP_m{}'' - P_mP_n{}''$$

Therefore, the above expression reduces to

$$\left(1-x^2\right)\frac{d}{dx}\left(P_nP_m{}' - P_mP_n{}'\right) - 2x\left(P_nP_m{}' - P_mP_n{}'\right) = \left[n(n+1) - m(m+1)\right]P_mP_n$$

Focusing again on the left hand side of this last expression, we see that

$$\frac{d}{dx}\left\{\left(1-x^2\right)\left(P_nP_m{}' - P_mP_n{}'\right)\right\} = \left(1-x^2\right)\frac{d}{dx}\left(P_nP_m{}' - P_mP_n{}'\right) - 2x\left(P_nP_m{}' - P_mP_n{}'\right)$$

Therefore,

$$\frac{d}{dx}\left\{\left(1-x^2\right)\left(P_nP_m{}' - P_mP_n{}'\right)\right\} = \left[n(n+1) - m(m+1)\right]P_mP_n \tag{8.30}$$

Finally, noting that the LHS is now an exact differential, we can integrate this expression over the domain of interest to give

$$\left(1-x^2\right)\left(P_nP_m{}' - P_mP_n{}'\right)\Big|_{-1}^{1} = 0 = \left[n(n+1) - m(m+1)\right]\int_{-1}^{1} P_m(x)P_n(x)dx$$

Note that the first part of this expression vanishes because the $\left(1-x^2\right)$ term evaluated at the limits goes identically to zero. Thus, since $m \neq n$, the above expression reduces to

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \tag{8.31}$$

This is a statement of orthogonality for $m \neq n$.

Developing a general expression for the normalization (i.e. for the case where m = n) is not very straightforward at all and there are a number of approaches that can be used (all of which are tedious). The approach chosen here starts with the Generating Function for Legendre polynomials (mostly so we can show an example of its use),

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{8.32}$$

Squaring both sides of eqn. (8.32) gives

$$\frac{1}{1-2xt+t^2} = \left(\sum_{m=0}^{\infty} P_m(x)t^m\right)\left(\sum_{n=0}^{\infty} P_n(x)t^n\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)P_n(x)t^{m+n}$$

Now integrating this expression gives

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x)P_n(x)dx\right\}t^{m+n} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n{}^2(x)dx\right\}t^{2n}$$

where the last equality is a result of the orthogonality relationship in eqn. (8.31).

Working on the left hand side of this expression, we have with

$$a = 1 + t^2 \qquad z = \text{-}2tx \qquad \text{and} \qquad dz = -2tdx$$

that

$$\int_{+2t}^{-2t} -\frac{1}{2t}\frac{dz}{a+z} = -\frac{1}{2t}\ln(a+z)\Big|_{+2t}^{-2t} = -\frac{1}{2t}\ln\left(\frac{a-2t}{a+2t}\right)$$

$$= -\frac{1}{2t}\ln\left(\frac{1-2t+t^2}{1+2t+t^2}\right) = -\frac{1}{2t}\ln\left[\left(\frac{1-t}{1+t}\right)^2\right] = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

But for $t^2 < 1$, the term containing the natural log function can be rewritten in terms of an infinite series expansion as

$$\ln\left(\frac{1+t}{1-t}\right) = 2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \cdots\right) \tag{8.33}$$

Therefore, the integral becomes

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}$$

Finally, we have the result

$$2\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n^{\,2}(x)dx\right\}t^{2n}$$

and equating like coefficients, we see that

$$\int_{-1}^{1} P_n^{\,2}(x)dx = \frac{2}{2n+1} \tag{8.34}$$

which is the desired normalization for the orthogonality relation for Legendre polynomials when m = n.

The Matlab *legendre* Function

The above manipulations illustrate several features associated with Legendre polynomials, in particular, and more generally, a set of similar manipulations and relationships apply to all orthogonal polynomials. For practical use, however, a key feature is to have access to a set of appropriate computational tools that implement the important relationships needed in applications. Matlab indeed has a variety of "special function" functions (see ***help specfun***) and, in particular, a function file for evaluating Legendre polynomials, ***legendre***, is available. Actually, this Matlab function evaluates the associated Legendre polynomial, $P_n^m(x)$, but, as noted previously, this reduces to the standard Legendre polynomials for m = 0.

As a simple example of using Matlab's ***legendre*** function, the first six Legendre polynomials, P_0 – P_5, are evaluated within **lpoly_demo1.m** (see Table 8.1) and plotted in Fig. 8.1 A few things to note here are that:

1. All the functions evaluated at x = 1 yield a value of unity. This was expected since eqn. (8.19) for the "standard form" of the Legendre polynomials forces this normalization.
2. All the functions [except $P_0(x)$] have both positive and negative components, with n zero crossings for $P_n(x)$. This, of course, is consistent with the fact that there are n roots to an n^{th} order polynomial. Here we simply have n real roots in the range $-1 < x < 1$.

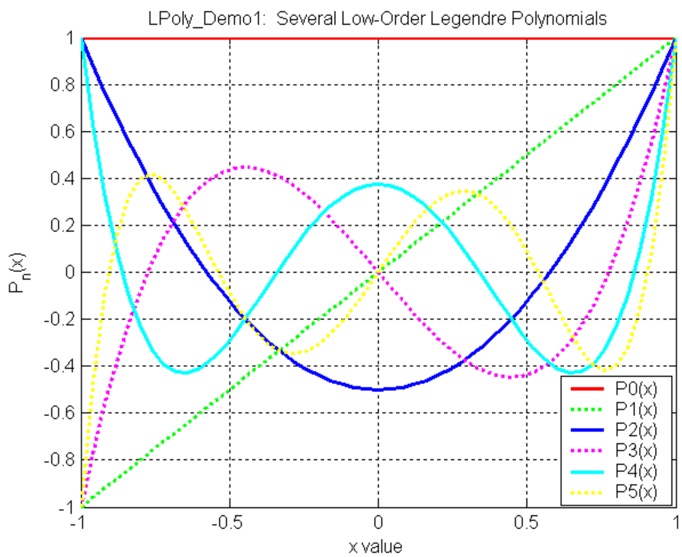

Fig. 8.1 Several low-order Legendre polynomials from lpoly_demo1.m.

We note also that the positive and negative behavior of the Legendre polynomials over the interval [-1,1] is essential for an orthogonality property to be valid. In fact, we demonstrate within **lpoly_demo1.m** that the first six Legendre polynomials do indeed satisfy the orthogonality relationships given in eqns. (8.31) and (8.34). In particular, the 6x6 table of integrals,

$$\int_{-1}^{1} P_m(x)P_n(x)dx \qquad \text{for } m = 0:5 \quad \text{and} \quad n = 0:5$$

are tabulated below (as produced from **lpoly_demo1.m** with the help of Matlab's ***quadl*** numerical integration capability):

n	m -> 0	1	2	3	4	5
0	2.000e+000	-1.388e-017	4.163e-017	6.939e-018	4.857e-017	0.000e+000
1	-1.388e-017	6.667e-001	-2.776e-017	-8.327e-017	0.000e+000	1.388e-017
2	4.163e-017	-2.776e-017	4.000e-001	0.000e+000	1.388e-017	0.000e+000
3	6.939e-018	-8.327e-017	0.000e+000	2.857e-001	0.000e+000	1.388e-017
4	4.857e-017	0.000e+000	1.388e-017	0.000e+000	2.222e-001	0.000e+000
5	0.000e+000	1.388e-017	0.000e+000	1.388e-017	0.000e+000	1.818e-001

Notice that all the off-diagonal elements in the 6x6 matrix are very small (essentially zero relative to the diagonal elements), and the diagonal entries do indeed satisfy the 2/(2n+1) normalization associated with the Legendre polynomials. The data presented here give some validation of Matlab's ***legendre*** function but, more importantly (since I never had any doubt about Matlab's accuracy or robustness), this demo simply shows that the capability exists and that it is pretty simple to use in a variety of situations…

Table 8.1 Listing of Matlab files lpoly_demo1.m and lpoly_demo1a.m.

```
%
%   LPOLY_DEMO1.M    Sample script file to plot several low-order Legendre
%                  polynomials and to demonstrate their orthogonality property
%
%   This sample file generates plots of the P0(x) - P5(x) Legendre polynomials.  It
%   also uses Matlab's QUADL routine to evaluate integrals of Pm(x)Pn(x) over the
%   interval [-1,1].  This should show the orthogonality property of the Legendre
%   polynomials.
%
%   The real purpose here is simply to demonstrate the use of Matlab's LEGENDRE
%   function.
%
%   File written by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all, close all,  nfig = 0;
%
%   set color and marker code for creating plots
      Ncm = 6;
      scm = ['r-';      % red solid
             'g:';      % green dotted
             'b-';      % blue solid
             'm:';      % magenta dotted
             'c-';      % cyan solid
             'y:'];     % yellow dotted
%
%   set up independent variable
      Nx = 51;   x = linspace(-1,1,Nx);
%
%   evaluate Pn(x) for n = 0:5
%   (note that m = 0 in the associated Legendre polynomials gives the desired
%    functions, and this is the first row of the variable returned from LEGENDRE)
      P = zeros(Nx,6);     % initialize space for storage of Pn(x)
      for n = 1:6
        AP = legendre(n-1,x);   P(:,n) = AP(1,:)';
      end
%
%   now let's plot all six curves
      nfig = nfig+1;  figure(nfig)
      for n = 1:6
        plot(x,P(:,n),scm(n,:),'LineWidth',2), grid on, hold on
        txt(n) = {['P',num2str(n-1),'(x)']};
      end
      title('LPoly\_Demo1:  Several Low-Order Legendre Polynomials')
      xlabel('x value'),ylabel('P_n(x)')
      legend(txt)
%
%   evaluate the orthogonality condition (use Matlab's QUADL routine)
      PmPn = zeros(6,6);   % initialize space for storage of PmPn integrals
      for n = 1:6
        for m = 1:6
          PmPn(m,n) = quadl('lpoly_demo1a',-1,1,[],[],m,n);
        end
      end
%
%   print out table of PmPn integrals
      fprintf('   n   m ->   0             1              2              3              4
5 \n')
      for n = 1:6
        fprintf(' %3i   %12.3e   %12.3e   %12.3e   %12.3e   %12.3e   %12.3e \n', ...
                 n-1,PmPn(:,n));
      end
%
%   end of demo
```

```
%
%   LPOLY_DEMO1A.M   Called by QUADL to determine Pm*Pn integrals
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%
      function f = ifile(x,M,N)
      APm = legendre(M-1,x);   APn = legendre(N-1,x);
      f = APm(1,:).*APn(1,:);   % recall that the 1st row is the Legendre poly
%
%   end of function
```

Application Notes

The primary purpose of the above developments is simply to demonstrate several important relations for a particular set of orthogonal polynomials. Similar manipulations can be performed for the other orthogonal functions (Hermite polynomials, Laguerre polynomials, etc.) and the reader is encouraged to seek out further details as needed for a particular application. Note that the choice of the specific orthogonal polynomial for a given application is often dictated by the domain of interest.

For Legendre polynomials, for example, the functions are orthogonal over an interval $-1 \le x \le +1$ and this range makes them particularly suitable for problems involving spherical coordinates. In particular, Legendre polynomials are used extensively where the directional dependence of some quantity is treated explicitly -- such as particle transport problems. Often, one of the direction variables, say θ, ranges from 0 to π (i.e. $0 \le \theta \le \pi$) and a simple change of variables, $\mu = \cos\theta$, has μ varying between ± 1; the domain of interest for Legendre polynomials.

For example, say some quantity, Σ, is a function of the direction variable θ. Then,

$$\Sigma(\theta) \to \Sigma(\cos\theta) \to \Sigma(\mu) \qquad \text{where} \qquad \mu = \cos\theta$$

and one can write $\Sigma(\mu)$ in terms of Legendre polynomials, or

$$\Sigma(\mu) = \sum_{n=0}^{\infty} a_n P_n(\mu) \approx \sum_{n=0}^{N} a_n P_n(\mu) \tag{8.35}$$

where, in practice, the infinite series is truncated to a finite number of terms, giving an approximate relationship for $\Sigma(\mu)$. The first part of eqn. (8.35) is just a Generalized Fourier Series (or sometimes called a Fourier-Legendre series) representation for the function $\Sigma(\mu)$. The truncation to a finite number of terms represents the usual approximation made in most practical applications.

The expansion coefficients in eqn. (8.35) can be found by multiplying both sides of the expression by $P_m(\mu)$ and integrating to give

$$\int_{-1}^{1} P_m(\mu)\Sigma(\mu)d\mu = \sum_{n=0}^{N} a_n \int_{-1}^{1} P_m(\mu)P_n(\mu)d\mu$$

Finally, one simply uses the orthogonality property of the Legendre polynomials and solves for a_n, which gives

$$\int_{-1}^{1} P_m(\mu)\Sigma(\mu)d\mu = \sum_{n=0}^{N} a_n \left(\frac{2}{2n+1}\right)\delta_{mn} = a_m \left(\frac{2}{2m+1}\right)$$

or

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(\mu)\Sigma(\mu)d\mu \tag{8.36}$$

One computes and stores the a_n's given basic information about $\Sigma(\mu)$, and then, when needed, $\Sigma(\mu)$ is reconstructed using eqn. (8.35).

Note: In neutron and photon transport analyses, $\Sigma(\mu)$ is the macroscopic scattering cross section as a function of the scattering angle (a ***cross section*** is related to the probability that a particular interaction will occur). This quantity is computed, on the fly, as part of the transport computations, and it is the expansion coefficients that are actually stored in the cross section library used in the code calculations. Most discrete ordinates transport codes (like the ANISN or DORT codes, for example) use a low order expansion for $\Sigma(\mu)$ (i.e. N = 3 or 5). For example, if N = 5 in a given calculation, we refer to the cross section representation as a P_5 approximation (which implies that a set of Legendre polynomials up to 5^{th} order are used to represent the functional dependence of the cross sections with scattering angle). This approach gives good results and it saves a considerable amount of computational time and memory (relative to the use of the exact $\Sigma(\mu)$ behavior of each material).

Bessel's Equation and Bessel Functions (in more detail)

Another important class of special functions is the so-called ***Bessel Functions***. These functions are applicable in a wide variety of situations and, similar to the other special functions, one particular set of Bessel functions also has the property of orthogonality. This subsection overviews the definition and development of the Bessel functions and highlights some key features that are useful in practical application.

Bessel's Equation

The ordinary Bessel equation is given as

$$x^2y'' + xy' + \left(\lambda^2x^2 - \nu^2\right)y = 0 \tag{8.37}$$

where ν is referred to as the order of the Bessel function and λ is a parameter within the argument of the resultant Bessel functions. If we let $t = \lambda x$, then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\lambda\frac{dy}{dt}\right)\frac{dt}{dx} = \lambda^2\frac{d^2y}{dt^2}$$

Therefore, with these substitutions, eqn. (8.37) becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \left(t^2 - \nu^2\right)y = 0 \tag{8.38}$$

This form, written with t = x, gives

$$x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \tag{8.39}$$

This is the most common representation of Bessel's equation. This is unfortunate since eqn. (8.37) is more general and actually occurs more frequently in practice. However, as shown here, the extension to included a parameter λ is straightforward (we simply replace x with λx).

One Solution via the Power Series Method

Since eqn. (8.39) has a regular singular point at x = 0, we need to use the extended power series method. Thus, we try

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \tag{8.40}$$

and, upon substitution of this assumed solution and its derivatives into the defining differential equation, the indicial equation becomes (for $a_0 \neq 0$),

$$(r+\nu)(r-\nu) = 0 \tag{8.41}$$

Therefore, one gets two roots: $r_1 = \nu$ and $r_2 = -\nu$.

Focusing first on $r_1 = \nu$, the recurrence relation becomes

$$a_{2m} = -\frac{1}{2^2 m(\nu + m)} a_{2m-2} \tag{8.42}$$

for $m = 1, 2, 3, \cdots$. This form is a little different than usual. In particular, since in the typical representation all the odd coefficients vanish (i.e. $a_1, a_3, a_5, \cdots = 0$), we simply replaced m with 2m-2, which then reduces to eqn. (8.42), with the index m varying from 1 to ∞ in unit increments (this is why the above coefficient is written as a_{2m}).

Finally, to put the solution into standard form, we define a_0 as

$$a_0 = \frac{1}{2^\nu \nu!} \tag{8.43}$$

and the first solution to the ordinary Bessel's equation becomes

$$y_1(x) = \sum_{m=0} a_{2m} x^{2m+\nu} \tag{8.44}$$

This function is called an ordinary Bessel function of the first kind and it is denoted by $J_\nu(x)$. After some manipulation, the infinite series representation for $J_\nu(x)$ can be written as

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(m + \nu + 1)} \tag{8.45}$$

where $\Gamma(n)$ is the generalized factorial function (i.e. the ***Gamma Function***). Equation (8.45) is the formal definition of $J_\nu(x)$ and this series converges for all values of x.

Performing similar operations for $r_2 = -\nu$ gives a second solution to eqn. (8.39), or

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m - \nu + 1)} \tag{8.46}$$

This series converges for all values of x except for x = 0.

Linear Independence

If ν is not an integer, J_ν and $J_{-\nu}$ are linearly independent. We can see this by focusing on the functions in the vicinity of x = 0. Near x = 0, the negative exponent in $J_{-\nu}$ indicates that this function is unbounded, while J_ν is clearly bounded. Therefore, the two functions are not proportional - thus they must be linearly independent.

For ν equal to an integer, the situation is quite different. In this case the two roots of the indicial equation from the power series solution differ by an integer, and we have learned to be cautious about linear independence when this occurs. To address the question of linear independence further, consider the following equality (taken from Problem 10.10 in the Schaum's Outline Series, ***Advanced Mathematics***),

$$J_\nu'(x)J_{-\nu}(x) - J_{-\nu}'(x)J_\nu(x) = \frac{2\sin\nu\pi}{\pi x} \tag{8.47}$$

The left hand side of this relationship is simply the Wronskian of J_ν and $J_{-\nu}$ (with a negative sign), or

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_2'y_1 - y_1'y_2 = J_{-\nu}'J_\nu - J_\nu'J_{-\nu} \tag{8.48}$$

Now, for $\nu = n$ where n is an integer, the right hand side of eqn. (8.47) is clearly zero (i.e. $\sin n\pi = 0$ for integer n). Therefore, W = 0, and J_n and J_{-n} are linearly dependent. In fact, it is easy to show from the infinite series representations that, for n an integer,

$$J_{-n}(x) = (-1)^n J_n(x) \tag{8.49}$$

For ν not an integer, $\sin\nu\pi \neq 0$ and $W \neq 0$, and the two solutions, J_ν and $J_{-\nu}$, are linearly independent (as shown above). Therefore, when ν is not an integer the general solution to the ordinary Bessel's equation becomes

$$y(x) = c_0 J_\nu(x) + c_1 J_{-\nu}(x) \tag{8.50}$$

When $\nu = n$ is an integer, we need to develop a second linearly independent solution via some other means.

Ordinary Bessel Functions of the Second Kind

In searching for a second linearly independent solution, consider the following development. For $\nu \neq$ integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent and eqn. (8.50) represents the general solution to eqn. (8.39). Now let's define a new function, $Y_\nu(x)$, in terms of these two linearly independent functions, or

$$Y_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \tag{8.51}$$

Now since $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent for non-integer ν, we can write the general solution to Bessel's equation as

$$y(x) = A_0 J_\nu(x) + A_1 Y_\nu(x) \tag{8.52}$$

where it is easy to see the correspondence with eqn. (8.50) with values of c_0 and c_1 given by

$$c_0 = A_0 + A_1\frac{\cos\nu\pi}{\sin\nu\pi} \qquad \text{and} \qquad c_1 = -\frac{A_1}{\sin\nu\pi}$$

Now our real interest with these manipulations is to determine what happens when ν becomes an integer. For this situation, let's take the limit of eqn. (8.51) as $\nu \to n$. Performing this operation gives

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) = \lim_{\nu\to n}\left[\frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}\right] \tag{8.53}$$

which, via eqn. (8.49), gives an indeterminate form upon substitution, or

$$Y_n(x) = \frac{(-1)^n J_n(x) - (-1)^n J_n(x)}{\sin n\pi} = \frac{0}{0}$$

Therefore, to determine this limit, we use L'Hospital's Rule, or

$$Y_n(x) = \lim_{\nu \to n} \left[\frac{\cos \nu\pi \frac{d}{d\nu} J_\nu(x) - \pi \sin \nu\pi \, J_\nu(x) - \frac{d}{d\nu} J_{-\nu}(x)}{\pi \cos \nu\pi} \right]$$

where it is important to note that the derivative is taken with respect to ν. Upon actually taking the limit, we have

$$Y_n(x) = \frac{(-1)^n}{\pi} \left[(-1)^n \frac{d}{d\nu} J_\nu(x) - \frac{d}{d\nu} J_{-\nu}(x) \right]_{\nu=n} \tag{8.54}$$

Now taking the indicated derivatives, item by item, and simplifying, one gets (after considerable manipulation!!!)

$$Y_n(x) = \frac{2}{\pi} J_n(x) \left(\ln \frac{x}{2} + \gamma \right) + \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1} \left(h_m + h_{m+n} \right)}{2^{2m+n} m!(m+n)!} x^{2m} - \frac{x^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n} m!} x^{2m} \tag{8.55}$$

with $h_0 = 0$ $\quad h_s = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}$ $\quad$ and $\quad \gamma = \lim_{s \to \infty} \left(h_s - \ln s \right)$

where $\gamma \approx 0.577215665$ is known as Euler's constant. Although this function is very ugly and extremely tedious to work with in this form, it is, nevertheless, as important function. It is well known and it can be manipulated, evaluated numerically, plotted, differentiated, integrated, etc., just like any other function. $Y_\nu(x)$ and $Y_n(x)$ are known as ordinary Bessel functions of the second kind.

For numerical evaluation, the ordinary Bessel functions of the first and second kind are usually fit to polynomial expansions, the expansion coefficients are tabulated (see Abramowitz and Stegun's ***Handbook on Mathematical Functions***, for example), and a relatively simple polynomial is then evaluated each time one needs to compute $J_\nu(x)$ or $Y_\nu(x)$.

Summary Expressions -- Ordinary Bessel Functions

Ordinary Bessel Equation	$x^2 y'' + xy' + \left(\lambda^2 x^2 - \nu^2 \right) y = 0$
General Solution (ordinary)	$y(x) = A_0 J_\nu(\lambda x) + A_1 Y_\nu(\lambda x)$
Definition of Y_ν	$Y_\nu(\lambda x) = \frac{\cos \nu\pi \, J_\nu(\lambda x) - J_{-\nu}(\lambda x)}{\sin \nu\pi}$
Definition of Y_n	$Y_n(\lambda x) = \lim_{\nu \to n} Y_\nu(\lambda x)$

Summary Expressions -- Modified Bessel Functions

Modified Bessel Equation	$x^2y'' + xy' - \left(\lambda^2x^2 + \nu^2\right)y = 0$
General Solution (modified)	$y(x) = A_0 I_\nu(\lambda x) + A_1 K_\nu(\lambda x)$
Definition of I_ν	$I_\nu(\lambda x) = i^{-\nu} J_\nu(i\lambda x)$
Definition of K_ν	$K_\nu(\lambda x) = \dfrac{\pi}{2} \dfrac{I_{-\nu}(\lambda x) - I_\nu(\lambda x)}{\sin \nu\pi}$
Definition of K_n	$K_n(\lambda x) = \lim\limits_{\nu \to n} K_\nu(\lambda x)$

In the above table, $I_\nu(\lambda x)$ is referred to as a modified Bessel function of the first kind and $K_\nu(\lambda x)$ is known as a modified Bessel function of the second kind. I_ν and K_ν are linearly independent for any $\nu > 0$.

Summary Expressions -- Hankel Functions

The Hankel functions of the first and second kind are complex conjugates and they are written as

$$H_\nu^{(1)}(\lambda x) = J_\nu(\lambda x) + iY_\nu(\lambda x) \qquad \text{and} \qquad H_\nu^{(2)}(\lambda x) = J_\nu(\lambda x) - iY_\nu(\lambda x) \tag{8.56}$$

Additional Properties and Relationships Among the Bessel Functions

Several important **Recurrence Formulas** (where we have not included the functional dependence on x for simplicity):

$J_{\nu+1} = \dfrac{2\nu}{x} J_\nu - J_{\nu-1}$	$Y_{\nu+1} = \dfrac{2\nu}{x} Y_\nu - Y_{\nu-1}$
$I_{\nu+1} = I_{\nu-1} - \dfrac{2\nu}{x} I_\nu$	$K_{\nu+1} = K_{\nu-1} + \dfrac{2\nu}{x} K_\nu$

Some important **Derivative Formulas**:

$J_\nu' = J_{\nu-1} - \dfrac{\nu}{x} J_\nu = -J_{\nu+1} + \dfrac{\nu}{x} J_\nu$	$Y_\nu' = Y_{\nu-1} - \dfrac{\nu}{x} Y_\nu = -Y_{\nu+1} + \dfrac{\nu}{x} Y_\nu$
$I_\nu' = I_{\nu-1} - \dfrac{\nu}{x} I_\nu = I_{\nu+1} + \dfrac{\nu}{x} I_\nu$	$K_\nu' = -K_{\nu-1} - \dfrac{\nu}{x} K_\nu = -K_{\nu+1} + \dfrac{\nu}{x} K_\nu$

Some important **Integral Formulas**:

One can use the derivative formulas to derive various integral relations. For example, the above expression for $J_\nu'(x)$, for $\nu = 0$, gives

$$J_0'(x) = -J_1(x)$$

Thus, from this relationship, we have

$$\int J_1(x)dx = -J_0(x) \tag{8.57}$$

Similarly, the expression for $J_\nu'(x)$, for ν = 1, gives

$$J_1'(x) = J_0(x) - \frac{1}{x}J_1(x) \qquad \text{or} \qquad xJ_1'(x) + J_1(x) = xJ_0(x)$$

The left hand side of the last expression can be written as the derivative of the $xJ_1(x)$ product, or

$$\frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x)$$

Therefore, integrating this expression gives

$$\int xJ_0(x)dx = xJ_1(x) \tag{8.58}$$

Some Plots and Limiting Values for the Low-Order Bessel Functions

It is important to have a feeling for the functional behavior of the Bessel functions for various values of the argument x. This is particularly true for the low-order integer Bessel functions since they occur so frequently in practical applications. To show this behavior, a short Matlab file called **bessplt.m** has been written to plot some low-order Bessel functions and the resultant plots are given in Fig. 8.2. From here it is obvious that the ordinary Bessel functions are oscillatory in nature and that the modified Bessel functions tend to look more like decaying and growing exponentials (this is a rough description only). A listing of **bessplt.m** is given in Table 8.2, and this can serve as an example of how to work with Bessel functions within the Matlab environment.

Also of interest here are the limiting values of the low-order integer Bessel functions on the interval $\left[0 \le x \le \infty\right]$. In particular, the limiting values can be summarized as follows:

	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
as $x \to 0$	1	0	$-\infty$	$-\infty$	1	0	∞	∞
as $x \to \infty$	oscillates	oscillates	oscillates	oscillates	∞	∞	0	0

These quantities are particularly useful in evaluating boundary conditions for BVPs which can be solved in terms of integer-order Bessel functions.

The are many more useful relationships for the Bessel functions that have not been tabulated here, and the student is encouraged to browse the literature for a more comprehensive treatise on this subject. We will return to the subject of orthogonality for the ordinary Bessel functions in a later section, and the next subsection gives a recipe for treating a variety of general variable coefficient second-order equations with Bessel function solutions. Beyond this, if the need arises, the reader can always find additional information on this important subject from a variety of sources (there is a lot out there on a wide variety of subjects involving Bessel functions).

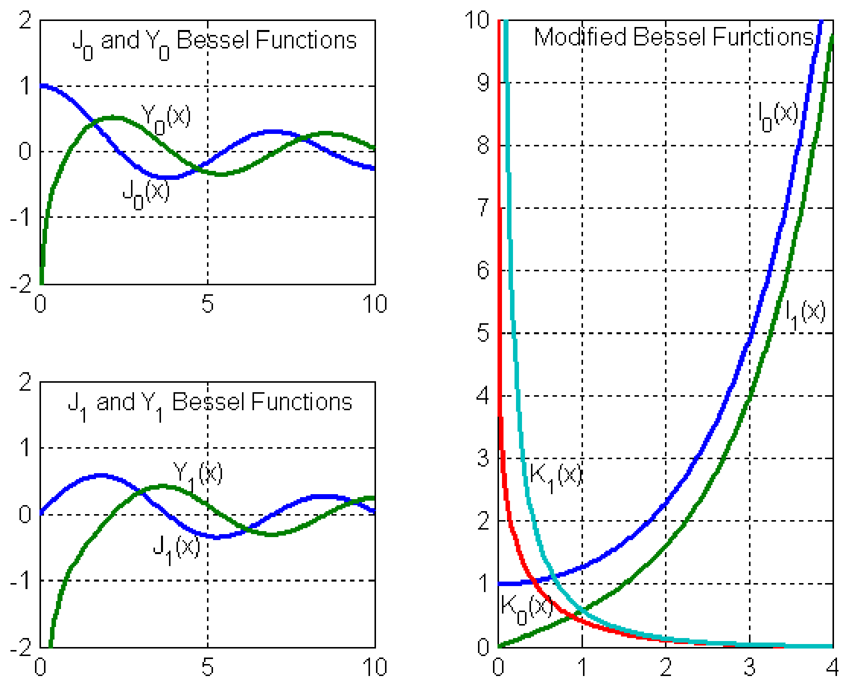

Fig. 8.2 Some plots for the low-order Bessel functions.

Table 8.2 Listing of Matlab m-file bessplt.m.

```
%
%   BESSPLT.M    Sample script file to plot some low-order Bessel Functions
%
%   This is a sample file to generate plots of the zero and first order Bessel
%   functions - J0(x), J1(x)    and   Y0(x), Y1(x)
%             - I0(x), I1(x)    and   K0(x), K1(x)
%
%   File written by J. R. White, UMass-Lowell (Aug. 2003)
%


%
%   getting started
      clear all, close all,   nfig = 0;
%
%   setup independent variable, but don't evaluate at exactly zero since some
%   of the functions have a singular point at zero
      Nx1 = 201;    x1 = linspace(eps,10,Nx1);    % range for ordinary BF plots
      Nx2 = 201;    x2 = linspace(eps,4,Nx2);     % range for modified BF plots
%
%   evaluate ordinary Bessel functions
      J0 = besselj(0,x1);     Y0 = bessely(0,x1);
      J1 = besselj(1,x1);     Y1 = bessely(1,x1);
%
%   evaluate modified Bessel functions
      I0 = besseli(0,x2);     K0 = besselk(0,x2);
      I1 = besseli(1,x2);     K1 = besselk(1,x2);
%
%   now let's plot these curves
      nfig = nfig+1;   figure(nfig)
      subplot(2,2,1),plot(x1,[J0;Y0],'LineWidth',2),grid
      axis([0 10 -2 2]);
      gtext('J_0(x)'),gtext('Y_0(x)')
      gtext('J_0 and Y_0 Bessel Functions')
```

```
%
      subplot(2,2,3),plot(x1,[J1;Y1],'LineWidth',2),grid
      axis([0 10 -2 2]);
      gtext('J_1(x)'),gtext('Y_1(x)')
      gtext('J_1 and Y_1 Bessel Functions')
%
      subplot(1,2,2),plot(x2,[I0;I1;K0;K1],'LineWidth',2),grid
      axis([0 4 0 10]);
      gtext('I_0(x)'),gtext('I_1(x)'),gtext('K_0(x)'),gtext('K_1(x)')
      gtext('Modified Bessel Functions')
%
%    end of demo
```

Equations Solvable in Terms of Bessel Functions

If $(1-a)^2 \geq 4c$ and d, p, q are nonzero, then the differential equation

$$x^2y'' + x\left(a + 2bx^p\right)y' + \left[c + dx^{2q} + b\left(a + p - 1\right)x^p + b^2x^{2p}\right]y = 0 \tag{8.59}$$

has complete solution

$$y(x) = x^\alpha e^{-\beta x^p}\left[C_1J_\nu\left(\lambda x^q\right) + C_2Y_\nu\left(\lambda x^q\right)\right] \tag{8.60}$$

where

$$\alpha = \frac{1-a}{2} \qquad \beta = \frac{b}{p} \qquad \lambda = \frac{|d|^{\frac{1}{2}}}{q} \qquad \nu = \frac{1}{2q}\left[(1-a)^2 - 4c\right]^{\frac{1}{2}} \tag{8.61}$$

with conditions:

1. if $d < 0$, replace J_ν and Y_ν with I_ν and K_ν
2. if $\nu \neq n$, Y_ν and K_ν can be replaced with $J_{-\nu}$ and $I_{-\nu}$

An exception to the above rule exists only when the equation reduces exactly to a second-order Euler-Cauchy equation of the form

$$\mathbf{x^2}y'' + axy' + by = 0$$

which has solutions in the form $y = x^m$ (see Section II of these notes).

The expressions summarized in eqns. (8.59) - (8.61) represent a recipe for analytically solving a wide class of problems in terms of the ordinary or modified Bessel functions. Many 2nd order variable coefficient linear systems can be cast into this form and, if this can be done, the above equations represent a systematic approach for solving these systems. Two specific examples, Example 8.1 and Example 8.2, illustrate the use of this general relationship.

A third problem involving Bessel functions, Example 8.3, is also available. This example illustrates the use of the modified Bessel functions to get an analytical solution to a simple heat conduction problem in cylindrical coordinates. Plotting of the resultant temperature and gradient profiles is done in a simple Matlab file as another example showing the evaluation of the Bessel functions within the Matlab environment.

Example 8.1 -- Solution Using Elementary and Bessel Function Methods

Problem Description:

Find the general solution to the following equation using an elementary approach (linear constant coefficient system) and by using the general form of Bessel's equation:

$$y'' + y = 0$$

Problem Solution:

Method 1 Elementary Solution

This is a constant coefficient linear 2nd order ODE. Therefore, we let $y = e^{\lambda x}$, and the characteristic equation becomes $\lambda^2 + 1 = 0$ with roots $\lambda = \pm i$. Thus the general solution can be written as

$$y(x) = k_1 e^{ix} + k_2 e^{-ix} \qquad \text{or} \qquad y(x) = c_1 \cos x + c_2 \sin x$$

Method 2 Bessel Function Solution

Comparing the defining ODE with the most general form of Bessel's equation [see eqns. (8.59) - (8.61)], we have

$$x^2 y'' + x^2 y = 0$$

By equating the coefficient of the $y'(x)$ term (i.e. $a + 2bx^p = 0$), we have $a = b = 0$. Therefore, the coefficient of the $y(x)$ term becomes

$$c + dx^{2q} + b(a + p - 1)x^p + b^2 x^{2p} = x^2$$

but with $a = b = 0$, we have

$$c + dx^{2q} = x^2$$

Therefore, letting $c = 0$, $d = 1$, and $q = 1$ gives the desired equality.

With all the coefficients known and all the proper conditions satisfied, we can evaluate the constants in the general solution as follows:

$$\alpha = \frac{1-a}{2} = \frac{1}{2} \qquad \beta = \frac{b}{p} = 0$$

$$\lambda = \frac{\sqrt{|d|}}{q} = 1 \qquad \nu = \frac{\sqrt{(1-a)^2 - 4c}}{2q} = \frac{1}{2}$$

Therefore, the solution to the original ODE becomes

$$y(x) = x^{1/2}\left[a_1 J_{1/2}(x) + a_2 J_{-1/2}(x)\right]$$

This implies that the half-order Bessel functions must be related to the sine and cosine functions, since the solutions using the two different methods must be identical. From Problem 10.4 in the Schaum's Outline Series on ***Advanced Mathematics,*** we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad \text{and} \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Therefore the solution for y(x) can be written as

$$y(x) = \sqrt{x}\left[a_1 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \sin x + a_2 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \cos x \right] = c_1 \sin x + c_2 \cos x$$

and this solution is of the form that we expect for a linear constant coefficient system. Although using Bessel functions is not the most efficient way to go for this problem, this example simply illustrates that the Bessel functions are applicable to a wide variety of systems (they were originally identified as a set of solutions for variable coefficient systems and here they were used to solve a constant coefficient system).

Example 8.2 -- Solution Using Bessel Function Methods

Problem Description:

Find the general solution to the following equation using the general form of Bessel's equation:

$$x^2 y'' + x\left(4x^4 - 3\right)y' + \left(4x^8 - 5x^2 + 3\right)y = 0$$

Problem Solution:

By equating the coefficients of the $y'(x)$ term with the most general form of Bessel's equation [see eqns. (8.59) - (8.61)], we have

$$a + 2bx^p = -3 + 4x^4$$

Therefore, a = -3, b = 2, and p = 4.

With these constants specified, the coefficient for the y(x) term becomes

$$c + dx^{2q} + b\left(a + p - 1\right)x^p + b^2 x^{2p} = 3 - 5x^2 + 4x^8$$

$$c + dx^{2q} + 2(-3 + 4 - 1)x^4 + 4x^8 = 3 - 5x^2 + 4x^8$$

Therefore, c = 3, d = -5, and q = 1.

Now, since the conditions of the method are satisfied, the constants within the general solution become

$$\alpha = \frac{1-a}{2} = 2 \qquad \beta = \frac{b}{p} = \frac{1}{2}$$

$$\lambda = \frac{\sqrt{|d|}}{q} = \sqrt{5} \qquad \nu = \frac{\sqrt{(1-a)^2 - 4c}}{2q} = \frac{\sqrt{16-12}}{2} = 1$$

Finally, since d < 0, we have

$$y(x) = x^2 e^{-\frac{x^4}{2}}\left[c_1 I_1\left(\sqrt{5}x\right) + c_2 K_1\left(\sqrt{5}x\right)\right]$$

This represents an analytical solution to the given problem (a tough problem indeed). In a realistic BVP, one would now apply appropriate boundary conditions to uniquely identify the two arbitrary coefficients within the general solution. The next example takes this final step to produce a complete unique solution to a particular heat transfer application.

Example 8.3 -- Analytical Solution to the Circular Fin Problem

Problem Description:

With the figure, general notation, and the model development given previously (see Section V), analytically determine the temperature and temperature gradient profiles for the circular fin problem given the following numerical data:

r_w = 1 in.	r_s = 1.5 in.	δ = 0.0625 in.
T_w = 200 °F	T_∞ = 70 °F	
h = 20 BTU/hr-ft^2-°F	k = 75 BTU/hr-ft-°F	

Evaluate and plot the normalized temperature and gradient profiles and determine the absolute fin edge temperature. Also determine the total heat loss from the fin and compute the fin efficiency, η, where

$$\eta = \frac{\text{actual heat transfer}}{\text{heat transfer if entire fin is at } T_w}$$

Problem Solution:

The dimensionless form of the steady state energy balance for combined heat conduction and convection for a cylindrical fin configuration is given as (see the formal development with appropriate limitations and definitions in Section V of these notes):

$$x^2u'' + xu' - \alpha^2x^2u = 0 \qquad \text{with} \qquad \alpha^2 = \frac{2hr_s^2}{k\delta}$$

with boundary conditions,

$$\text{at } x = r_w/r_s = a, \quad u(a) = 1 \qquad \text{and} \qquad \text{at } x = r_s/r_s = b = 1, \quad u'(b) = 0$$

With our recent discussion concerning Bessel functions, one recognizes this as a special form of the modified Bessel's equation,

$$x^2y'' + xy' - (\alpha^2x^2 - \nu^2)y = 0$$

with general solution

$$y(x) = c_1I_\nu(\alpha x) + c_2K_\nu(\alpha x)$$

Comparing this standard system to the energy balance for the problem of interest shows that $\nu = 0$ and the general solution for the normalized temperature profile is

$$u(x) = c_1I_0(\alpha x) + c_2K_0(\alpha x)$$

Applying the boundary conditions to this general solution gives:

1. For u(a) = 1, we have

$$c_1I_0(\alpha a) + c_2K_0(\alpha a) = 1$$

2. For $u'(b) = u'(1) = 0$, we have

$$u'(x) = c_1 \frac{d}{d(\alpha x)} I_0(\alpha x) \frac{d}{dx}(\alpha x) + c_2 \frac{d}{d(\alpha x)} K_0(\alpha x) \frac{d}{dx}(\alpha x)$$
$$= c_1 \alpha I_1(\alpha x) - c_2 \alpha K_1(\alpha x)$$

and letting x = 1 gives

$$c_1 \alpha I_1(\alpha) - c_2 \alpha K_1(\alpha) = 0$$

Thus, the two boundary conditions give two coupled equations for c_1 and c_2, or

$$c_1 I_0(\alpha a) + c_2 K_0\ (\alpha a) = 1$$
$$c_1 \alpha I_1(\alpha) - c_2 \alpha K_1(\alpha) = 0$$

From the second equation, we see that

$$c_1 = c_2 \frac{K_1(\alpha)}{I_1(\alpha)}$$

and putting this into the first equation in the set gives

$$c_2 \left[\frac{K_1(\alpha)}{I_1(\alpha)} I_0(\alpha a) + K_0(\alpha a) \right] = 1$$

Thus, the two coefficients become

$$c_2 = \frac{I_1(\alpha)}{K_1(\alpha) I_0(\alpha a) + I_1(\alpha) K_0(\alpha a)}$$

and

$$c_1 = \frac{K_1(\alpha)}{K_1(\alpha) I_0(\alpha a) + I_1(\alpha) K_0(\alpha a)}$$

Putting these constants into the general solution gives an explicit formulation for the normalized temperature profile, or

$$u(x) = \frac{K_1(\alpha) I_0(\alpha x) + I_1(\alpha) K_0(\alpha x)}{K_1(\alpha) I_0(\alpha a) + I_1(\alpha) K_0(\alpha a)}$$

The temperature gradient can also be evaluated to give

$$u'(x) = \frac{\alpha K_1(\alpha) I_1(\alpha x) - \alpha I_1(\alpha) K_1(\alpha x)}{K_1(\alpha) I_0(\alpha a) + I_1(\alpha) K_0(\alpha a)}$$

These analytical expressions are evaluated using the parameter specifications given above within the Matlab file **cylfina.m**. This file is listed in Table 8.3 and the resultant temperature and gradient profiles are plotted in Fig. 8.3. Note that the results here are exactly the same as those developed using the numerical techniques discussed in Section V (see Example 5.3A and Example 5.3B). As before, we also compute the fin efficiency as

$$\eta = \frac{q_{actual}}{q_{ideal}}$$

where the ideal energy transfer is computed assuming that the fin temperature is constant at the wall value, or

$$q_{ideal} = hA_c(T_w - T_\infty) = 2\pi h\left(r_s^2 - r_w^2\right)(T_w - T_\infty)$$

The actual energy transferred from the fin can be computed from the conduction representation at the wall, or

$$q_{actual} = -k\left(2\pi r_w \delta\right)\left.\frac{dT}{dr}\right|_{r_w} = -k\left(2\pi r_w \delta\right)\left(\frac{T_w - T_\infty}{r_s}\right)u'(a)$$

Table 8.4, which contains a listing of the output file from **cylfina.m**, shows that the numerical values for the heat transfer and fin efficiency from the analytical solution are exactly as computed in Example 5.3A and Example 5.3B (using numerical methods). The overall efficiency of about 93% and a tip temperature of almost 188 F - a drop of only 12 F from the wall temperature - indicate a fairly efficient overall fin arrangement.

This problem represents a good illustration of the use of Bessel functions in finding analytical solutions to a real problem. It also serves as a good example of how to apply the general analytical solution scheme, including the evaluation of the boundary conditions to determine the unknown coefficients in the general solution. In addition, the Matlab m-file associated with this problem, **cylfina.m**, can be used as another example of the evaluation of the Bessel functions within the Matlab environment. Finally, the combination of this example and those given in Section V (Example 5.3A and Example 5.3B) also represents a series of applications that contrast the various techniques generally available for solving linear boundary value problems (BVPs).

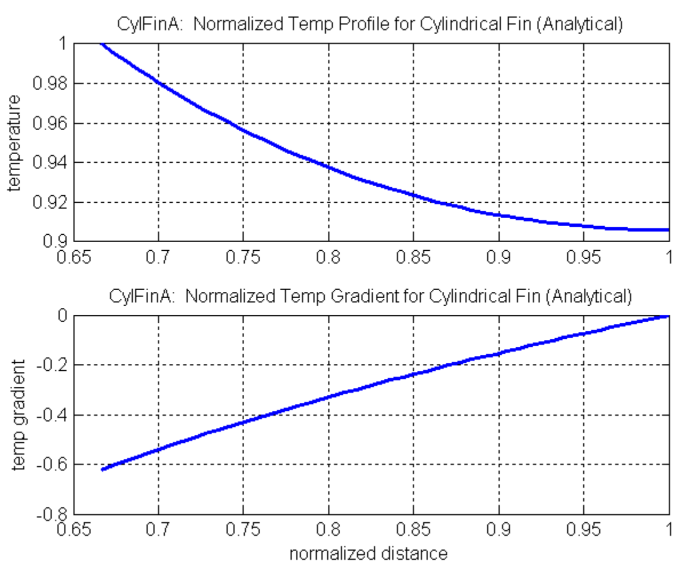

Fig. 8.3 Solution profiles for the circular fin problem (Analytical Solution).

Table 8.3 Listing of Matlab program cylfina.m.

```
%
%   CYLFINA.M   Heat Transfer in a Cylindrical Fin (Analytical Solution)
%
%   This file solves the cylindrical fin heat transfer problem using modified
%   Bessel functions. The base problem is defined via the following equation:
%
%     x^2*u'' + x*u' - ALF2*x^2*u = 0    where   ALF2 = 2*h*rs^2/[k*thk]
%
%     with B.C.   u(a) = 1    and    u'(b) = 0
%      and        a = rw/rs   and    b = rs/rs = 1
%
%     where   u = normalized temp = [T(r) - Tinf]/[Tw - Tinf]
%             x = normalized distance = r/rs
%
%     with   rw, rs = inside and outside radius of fin, respectively
%            thk = thickness of fin
%     and    h, k, Tw, and Tinf are all given quantities (fixed)
%
%   From the normalized solution we can construct absolute profiles (if desired):
%       T(r)  =  Tinf + u(x)[Tw - Tinf]
%       T'(r) =  u'(x)[Tw - Tinf]/rs
%
%   The above development is given as part of the course notes in the Math
%   Methods course (10/24.539).
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all, close all, nfig = 0;
%
%   basic data for the problem
      rw = 1/12;    rs = 1.5/12;    % inside and outside radius (ft)
      thk = .0625/12;               % thickness of fin (ft)
      Tw = 200;    Tinf = 70;       % inside wall and ambient temps (F)
      h = 20;                       % heat transfer coeff (BTU/hr-ft^2-F)
      k = 75;                       % thermal conductivity (BTU/hr-ft-F)
%
%   derived constants
      a = rw/rs;   b = rs/rs;
      alf2 = (2*h*rs*rs)/(k*thk);    alpha = sqrt(alf2);
      Qideal = 2*pi*h*(rs*rs-rw*rw)*(Tw-Tinf);
%
%   write base data to output file
      fid = fopen('cylfina.out','w');
      fprintf(fid,'\n *** CYLFINA.OUT ***  Data and Results from CYLFINA.M \n');
      fprintf(fid,'\n \nBASIC DATA FOR PROBLEM \n');
      fprintf(fid,'Inside and outside radius:  rw =  %6.2f ft \t rs =   %5.2f ft \n',rw,rs);
      fprintf(fid,'Thickness of fin:           thk = %6.2f ft \n',thk);
      fprintf(fid,'Inside/ambient temps:       Tw =  %6.2f F  \t Tinf = %5.2f F \n',Tw,Tinf);
      fprintf(fid,'Heat transfer coeff:        h =   %6.2f Btu/hr-ft^2-F \n',h);
      fprintf(fid,'Thermal conductivity:       k =   %6.2f Btu/hr-ft-F \n',k);
%
%   define solution domain
      x = linspace(a,b,50);       [nr,nc] = size(x);
%
%   evaluate constants in temperature equation
      a1 = besselk(1,alpha*b);      a2 = besseli(1,alpha*b);
      b1 = besseli(0,alpha*a);      b2 = besselk(0,alpha*a);
      denom = a1*b1+a2*b2;
%
%   create normalized temp and gradient
      u = (a1*besseli(0,alpha*x) + a2*besselk(0,alpha*x))/denom;
      up = alpha*(a1*besseli(1,alpha*x) - a2*besselk(1,alpha*x))/denom;
%
%   calc heat transferred (conduction) & fin eff. (edit key parameters)
      Qactual = -k*(2*pi*a*thk)*(Tw-Tinf)*up(1);
      Ttip = Tinf + (Tw-Tinf)*u(nc);
      fineff = Qactual/Qideal;
      fprintf(fid,'\n \nANALYTICAL SOLUTION RESULTS \n');
      fprintf(fid,'Wall Temp (F) =     %8.3f \n',Tw);
```

```
      fprintf(fid,'Ambient Temp (F) =  %8.3f \n',Tinf);
      fprintf(fid,'Tip Temp (F) =      %8.3f \n',Ttip);
      fprintf(fid,'Qideal (BTU/hr) =   %8.3f \n',Qideal);
      fprintf(fid,'Qactual (BTU/hr) =  %8.3f \n',Qactual);
      fprintf(fid,'Fin Eff =           %8.3f \n',fineff);
%
%   plot normalized profiles
      nfig = nfig+1;    figure(nfig)
      subplot(2,1,1)
      plot(x,u,'LineWidth',2),grid
      title('CylFinA:  Normalized Temp Profile for Cylindrical Fin (Analytical)')
      ylabel('temperature')
      subplot(2,1,2)
      plot(x,up,'LineWidth',2),grid
      title('CylFinA:  Normalized Temp Gradient for Cylindrical Fin (Analytical)')
      xlabel('normalized distance'),ylabel('temp gradient')
%
%   close output file
      fclose(fid);
%
%   end simulation %
```

Table 8.4 Listing of the output file for Example 8.3.

```
 *** CYLFINA.OUT ***  Data and Results from CYLFINA.M


BASIC DATA FOR PROBLEM
Inside and outside radius:  rw =    0.08 ft   rs =    0.13 ft
Thickness of fin:           thk =   0.01 ft
Inside/ambient temps:       Tw =  200.00 F    Tinf = 70.00 F
Heat transfer coeff:        h =    20.00 Btu/hr-ft^2-F
Thermal conductivity:       k =    75.00 Btu/hr-ft-F


ANALYTICAL SOLUTION RESULTS
Wall Temp (F) =      200.000
Ambient Temp (F) =    70.000
Tip Temp (F) =       187.784
Qideal (BTU/hr) =    141.808
Qactual (BTU/hr) =   132.293
Fin Eff =              0.933
```