Mathematical Methods (10/24.539)

VII. Power Series Solution Method

Introduction

For second or higher order systems, except for some special cases, we have only been able to generate analytical solutions for linear constant coefficient systems. This section of notes highlights the ***Power Series*** solution scheme, which gives a fairly general procedure to handle linear variable coefficient systems. In practice, however, this method is very tedious to apply and the resultant infinite series solutions are often difficult to use in subsequent manipulations and analyses. Sometimes a closed form solution can be written, but this is a special case rather than the rule.

Although not the method of choice in most practical applications, the Power Series solution method is the primary tool for solving a wide range of classical second order systems which give rise to many of the special functions that are used routinely in engineering design and analysis. Thus, we will study this technique as a tool for solving some model second order variable coefficient systems, and as background so that the study of ***Special Functions*** can be addressed in a logical manner (in the next section of these notes).

After a brief theoretical overview, the material here on the Power Series method focuses on the illustration of the method via actual applications to three specific problems, as follows:

Overview of the Method

- Some Definitions
- Computer Evaluation of Infinite Power Series
- Theorem #1 - Power Series Solution
- Theorem #2 - Extended Power Series Solution
- Relationships for y' and y''
- Solution Outline

Examples of Analytical Solutions

- **Example 7.1 - Solve** $\left(1-x^2\right)y''-2xy'+2y=0$
- **Example 7.2 - Solve** $8x^2y''+10xy'+\left(x-1\right)y=0$
- **Example 7.3 - Solve** $\left(x^2-1\right)x^2y''-\left(x^2+1\right)xy'+\left(x^2+1\right)y=0$

Overview of the Method

Some Definitions

The Power Series method assumes homogeneous solutions of the form

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m \qquad \text{or} \qquad y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^{m+r} \tag{7.1}$$

where the a_m coefficients and the r exponent need to be determined as part of the solution procedure. As such, we need to work with infinite series of the form given in eqn. (7.1). Thus, we begin our discussion of the method with a few definitions establishing the proper terminology:

Analytic Function -- A function is said to be analytic at $x = x_o$ if it can be represented by a power series of $(x - x_o)$ with a radius of convergence $R > 0$.

Radius of Convergence -- R is the radius of convergence if a series converges for x in the range defined by $|x - x_o| < R$ and diverges for x in the range $|x - x_o| > R$.

If the function f(x) is analytic at $x = x_o$, then it can be written as

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m = a_0 + a_1 (x - x_o) + a_2 (x - x_o)^2 + \cdots \tag{7.2}$$

where the radius of convergence is given by

$$R = \frac{1}{\lim\limits_{m\to\infty} \sqrt[m]{a_m}} \qquad \textbf{or} \qquad R = \frac{1}{\lim\limits_{m\to\infty} \left| \dfrac{a_{m+1}}{a_m} \right|} \tag{7.3}$$

As an example, consider the following expansion for e^x around the point $x_o = 0$,

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots$$

Therefore,

$$a_m = \frac{1}{m!} \qquad \text{and} \qquad \left| \frac{a_{m+1}}{a_m} \right| = \frac{\dfrac{1}{(m+1)!}}{\dfrac{1}{m!}} = \frac{m!}{(m+1)m!} = \frac{1}{m+1}$$

and the radius of convergence is given by

$$R = \frac{1}{\lim\limits_{m\to\infty} \dfrac{1}{m+1}} = \infty$$

Thus this series converges for all values of x.

As another example, consider the following expansion,

$$f(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m}x^{3m}=1-\frac{x^3}{8}+\frac{x^6}{64}-\frac{x^9}{512}+\cdots$$

If we let $t=x^3$, then

$$f(t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m}t^m \quad \text{and} \quad \left|\frac{a_{m+1}}{a_m}\right|=\frac{8^m}{8^{m+1}}=\frac{8^m}{8\left(8^m\right)}=\frac{1}{8}$$

Therefore,

$$R=\frac{1}{\lim\limits_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|}=8$$

Thus this particular series converges for $|t|<8$ or for $|x|<2$.

Computer Evaluation of Infinite Power Series

Now, an important consideration is "How do we evaluate and plot infinite series in the form of eqn. (7.1)"? Obviously, we cannot use an infinite number of terms in the expansion. Thus, one concern is associated with determining how many terms to include. Clearly, if the series is convergent, the individual terms must eventually get smaller and smaller and approach zero as $m\to\infty$. Thus, one way to stop the summation is by setting some user-specified error tolerance, tol, and then one simply truncates the series when the relative change associated with adding another term is less than tol.

Another concern is computational efficiency, and a brute force evaluation of eqn. (7.1) could be very cumbersome if the number of terms in the series is large. One way to significantly improve the overall efficiency is to implement the infinite series as a recurrence relation of the form,

$$f(x)=\sum_{n=1}^{\infty}T_n(x) \tag{7.4}$$

with $$T_{n+1}=r_nT_n \tag{7.5}$$

where r_n is the ratio of the $(n+1)^{th}$ term, T_{n+1}, to the n^{th} term, T_n. With this form, one can easily and efficiently utilize the following algorithm:

Algorithm to Evaluate Infinite Power Series

1. Set maximum number of terms, maxT, and the user-specified tolerance, tol, for stopping the calculation.
2. Initialize the first term in the series -- set $T = T_1$.
3. Initialize partial sum after first term -- set $f = T$.
4. while mrerr > tol and n < maxT

    compute r, where $r_n = T_{n+1}/T_n$ (specific to function of interest)

```
    T = r*T                     (compute next term in series)
    f = f + T                   (update partial sum)
    mrerr = max(abs(T/f))       (compute maximum relative change due to (n+1)th term)
    n = n + 1                   (increment counter)
end
```

This algorithm works great, with only minor changes for just about any infinite series of interest! The only steps that are case-specific involve initializing the first term, T_1, and the computation of the ratio, $r_n = T_{n+1}/T_n$, whose formula must be determined prior to implementation. For example, for the function, f(x), given above, we have

$$r_n = \frac{T_{n+1}}{T_n} = \frac{(-1)^{n+1} x^{3(n+1)}}{8^{n+1}} \times \frac{8^n}{(-1)^n x^{3n}} = -\frac{1}{8} x^3$$

and the first term in the series is $T_1 = 1$.

Continuing this example, a Matlab program called **ps_demo1.m** was written to evaluate and plot this particular f(x). The program listing is given in Table 7.1 and a plot of f(x) over the range $0 < x < 1.8$ is shown in Fig. 7.1. The program logic is quite straightforward and it directly follows the algorithm given above. Over the range given, the function is well behaved and the series converges after 46 terms at each x value to within the specified tolerance of 10^{-6}. Note that this is a rather slowly convergent series, and this is not unusual for series that have a finite radius of convergence. Also note that since the radius of convergence for this problem is R = 2, as x approaches a value of 2, a continually increasing number of terms will be needed for convergence. If x > 2, then the series expansion simply diverges.

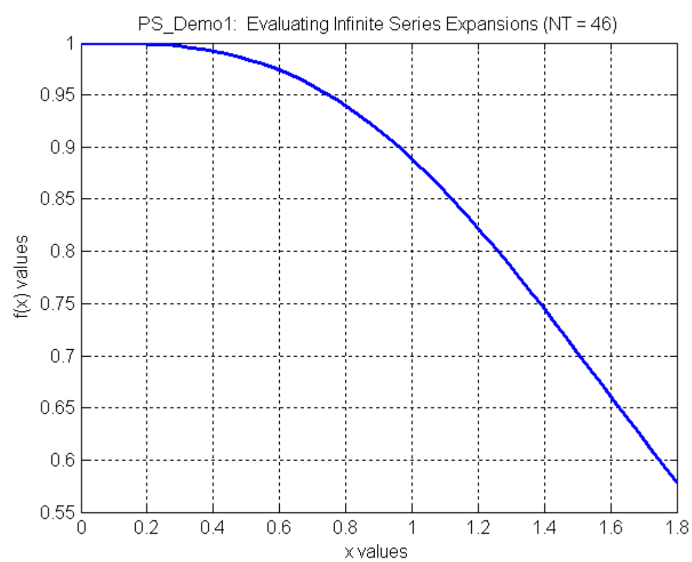

Fig. 7.1 Plot of f(x) from Matlab program ps_demo1.m.

Table 7.1 Listing of Matlab program ps_demo1.m.

```
%
%   PS_DEMO1.M   Evaluation of Infinite Power Series Expansions
%
%   This demo simply illustrates how to evaluate an infinite series
%   expansion for a particular function, f(x), as discussed in Section VII
%   of the Math Methods notes.  The key here is to write the series as a simple
%   recurrence relation, with term Tn+1 written as a function of term Tn (see
%   the notes for details).
%
%   File prepared by J. R. White, UMass-Lowell (Aug. 2003)
%

%
%   getting started
      clear all,  close all,   nfig = 0;
%
%   define range of x values (be sure to stay within the radius of convergence)
      x = linspace(0,1.8,50);
%
%   NOTE: For this problem the series converges for |x| < 2.  Just to see what happens
%   when you ignore this fact, you might try setting the maximum x to 2.1 (instead of 1.8).
%
%   loop over terms in expansion
      maxT = 101;  tol = 1e-6;   n = 0;  mrerr = 1.0;
      T = 1;  f = T;                   % initialize series
      while mrerr > tol  &  n < maxT
        r = - (1/8)*x.^3;              % compute r  (this is usually a function of n)
        T = r.*T;                      % compute Tn+1
        f = f + T;                     % add Tn+1 to partial sum
        i = find(f);                   % finds indices of nonzero values of f
        mrerr = max(abs(T(i)./f(i)));  % compute max relative error
        n = n+1;                       % increment counter
      end
      NT = n;                          % number of terms used in expansion
%
%   plot results
      nfig = nfig+1;   figure(nfig)
      plot(x,f,'b-','LineWidth',2), grid
      title(['PS\_Demo1:  Evaluating Infinite Series Expansions (NT = ',num2str(NT), ')']);
      xlabel('x values'),ylabel('f(x) values')
%
%   end of program
```

Theorem #1 -- Power Series Solution

If the functions p(x), q(x), and f(x) in the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{7.6}$$

are analytic at $x = x_o$, then every solution, y(x), is analytic at $x = x_o$ and it can be represented by a power series in powers of $x - x_o$ with radius of convergence $R > 0$. Therefore, we have a power series solution of the form given in the first part of eqn. (7.1), or

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m \tag{7.7}$$

Theorem #2 -- Extended Power Series Solution Method

The differential equation,

$$y''(x) + \frac{b(x)}{x - x_0} y'(x) + \frac{c(x)}{(x - x_0)^2} y(x) = f(x) \tag{7.8}$$

with b(x), c(x), and f(x) analytic around x_0 has ***at least one solution*** of the form given in the second part of eqn. (7.1), or

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^{m+r} \tag{7.9}$$

This is referred to as an extended power series with r chosen such that $a_0 \neq 0$. A second independent solution ***may*** contain a logarithmic term if the ***roots are repeated*** or if they ***differ by an integer***.

Relations for y' and y''

A function written in the form of a power series may be differentiated term by term. Therefore, based on the more general representation in eqn. (7.9), one has

$$y'(x) = \sum_{m=0}^{\infty} a_m (m + r)(x - x_0)^{m+r-1} \tag{7.10}$$

and

$$y''(x) = \sum_{m=0}^{\infty} a_m (m + r)(m + r - 1)(x - x_0)^{m+r-2} \tag{7.11}$$

and this list can be easily extended for higher-order derivatives.

Basic Solution Outline

Although somewhat tedious for most practical applications, a systematic procedure for the Power Series method can be identified, as follows:

1. Expand all the terms in the original differential equation in a power series about the point $x = x_0$ (in practice, $x_0 = 0$ in most cases).

2. Assume a solution of the form $y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^{m+r}$.

3. Substitute the assumed solution and its derivatives into the differential equation from Step 1 and shift all indices to that for the highest power of x with all the sums beginning with m = 0 **(this is a very important step)**.

4. Collect terms with like powers and equate the coefficients to the right hand side coefficients (these will be zero for a homogeneous equation).

5. Evaluate r in the original expression such that $a_0 \neq 0$ (this gives the indicial equation).

6. Obtain a recurrence relation for general term, a_m.

7. If repeated roots are obtained for r or $y_1(x)$ and $y_2(x)$ are linearly dependent, then one should use the ***variation of parameters (reduction of order) method*** to find a second linearly independent solution. In this case, simply let $y_2(x) = u(x)y_1(x)$. Note that, although conceptually straightforward, this can often be quite tedious if $y_1(x)$ cannot be written in a simple closed form solution.

This procedure is illustrated in the remainder of this section for a series of three cases. These illustrate the range of typical situations that arise in practical applications. The following section on Special Functions also gives some further examples of the basic Power Series solution methodology.

Example 7.1 -- Standard Power Series Solution

Problem Description:

Solve the following linear variable coefficient second order homogeneous system:

$$\left(1-x^2\right)y''-2xy'+2y=0$$

Problem Solution:

The original equation written in standard form is

$$y''-\frac{2x}{\left(1-x^2\right)}y'+\frac{2}{\left(1-x^2\right)}y=0$$

Since the coefficients are analytic at x = 0, we let

$$y(x)=\sum_m a_m x^m \qquad y'(x)=\sum_m a_m m x^{m-1} \qquad \text{and} \qquad y''(x)=\sum_m a_m m(m-1)x^{m-2}$$

Then, substitution into the original ODE gives

$$\sum_m a_m m(m-1)x^{m-2}-\sum_m a_m m(m-1)x^m-2\sum_m a_m m x^m+2\sum_m a_m x^m=0$$

Now, we can work on the first term to shift the exponent to the highest power (i.e. x^m), and also force the summation to begin at m = 0. Letting p = m-2 or m = p+2, we have

$$\sum_{m=0} a_m m(m-1)x^{m-2}=\sum_{p=-2} a_{p+2}(p+2)(p+1)x^p$$
$$=a_0(0)(-1)x^{-2}+a_1(1)(0)x^{-1}+\sum_{m=0} a_{m+2}(m+2)(m+1)x^m$$

where a_0 and a_1 are arbitrary coefficients because the coefficients of the x^{-2} and x^{-1} terms are already zero.

Substituting this result into the full balance equation gives

$$\sum_{m=0}\left\{a_{m+2}(m+2)(m+1)-a_m\left[m(m-1)+2m-2\right]\right\}x^m=0$$

or

$$a_{m+2}(m+2)(m+1)-a_m\left(m^2+m-2\right)=0$$

and

$$a_{m+2}=\frac{(m+2)(m-1)}{(m+2)(m+1)}a_m=\frac{(m-1)}{(m+1)}a_m$$

Therefore, letting m = 0, 1, 2, etc. gives

$$a_2 = -a_0 \qquad a_4 = \frac{1}{3}a_2 = -\frac{1}{3}a_0 \qquad a_6 = \frac{3}{5}a_4 = -\frac{1}{5}a_0$$

$$a_3 = 0 \qquad a_5 = \frac{2}{4}a_3 = 0 \qquad \text{etc.}$$

Therefore, the final solution, y(x), can be written as

$$y(x) = a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots\right) + a_1 x$$

where we have grouped all the terms that multiply the a_0 and a_1 coefficients separately. Here one recognizes two individual linearly independent solutions to the original ODE, or

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where $c_1 = a_0$ and $c_2 = a_1$, and

$$y_1(x) = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots \qquad \text{and} \qquad y_2(x) = x$$

Thus, the solution procedure is complete and the final result is represented as a linear combination of two linearly independent solutions, as expected.

Example 7.2 -- Extended Power Series Solution (Linearly Independent Solutions)

Problem Description:

Solve the following linear variable coefficient system:

$$8x^2y'' + 10xy' + (x-1)y = 0$$

Problem Solution:

The 2nd order ODE in standard form is

$$y'' + \frac{5/4}{x}y' + \frac{(x-1)/8}{x^2}y = 0$$

In this case, the full coefficients are not analytic at x = 0, but b(x) and c(x) are analytic in the general form,

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

Therefore, we need the extended power series representation for this problem, with

$$y(x) = \sum_m a_m x^{m+r}$$

and the corresponding derivative relationships

$$y'(x) = \sum_m a_m (m+r)x^{m+r-1} \qquad \text{and} \qquad y''(x) = \sum_m a_m (m+r)(m+r-1)x^{m+r-2}$$

Upon substitution, the original ODE becomes

$$8\sum_m a_m(m+r)(m+r-1)x^{m+r} + 10\sum_m a_m(m+r)x^{m+r} + \sum_m a_m x^{m+r+1} - \sum_m a_m x^{m+r} = 0$$

Combining all the coefficients for the terms with x^{m+r}, we let m = p+1 and p = m-1, giving

$$\sum_{p=-1}\left\{8a_{p+1}(p+r+1)(p+r) + 10a_{p+1}(p+r+1) - a_{p+1}\right\}x^{p+r+1} + \sum_{m=0} a_m x^{m+r+1} = 0$$

Now removing the p = -1 term from the sum gives

$$\left[8r(r-1) + 10r - 1\right]a_0 x^r + \sum_{m=0}\left\{\left[8(m+r+1)(m+r) + 10(m+r+1) - 1\right]a_{m+1} + a_m\right\}x^{m+r+1} = 0$$

Requiring that $a_0 \neq 0$ gives the indicial equation,

$$8r^2 + 2r - 1 = (4r-1)(2r+1) = 0$$

or

$$r_1 = \frac{1}{4} \qquad \text{and} \qquad r_2 = -\frac{1}{2}$$

with a_0 being an arbitrary constant. Note that these values for the roots of the indicial equation ***do not*** differ by an integer, so we expect that each value of r will lead to a linearly independent solution to the original ODE.

The desired recurrence relationship between a_{m+1} and a_m can be obtained by setting each coefficient of x^{m+r+1} to zero (because we have a homogeneous system), or

$$\sum_{m=0}\left\{\left[8(m+r+1)(m+r)+10(m+r+1)-1\right]a_{m+1}+a_m\right\}x^{m+r+1}=0$$

$$\sum_{m=0}\left\{\left[8m^2+16mr+18m+18r+8r^2+9\right]a_{m+1}+a_m\right\}x^{m+r+1}=0$$

and for $r_1 = 1/4$, this relationship gives

$$\sum_{m=0}\left\{\left[8m^2+4m+18m+\frac{9}{2}+\frac{1}{2}+9\right]a_{m+1}+a_m\right\}x^{m+r+1}=0$$

$$\sum_{m=0}\left\{\left[8m^2+22m+14\right]a_{m+1}+a_m\right\}x^{m+r+1}=\sum_{m=0}\left\{(4m+7)(2m+2)a_{m+1}+a_m\right\}x^{m+r+1}=0$$

Therefore, for $r_1 = 1/4$, we have,

$$a_{m+1}=-\left[\frac{1}{(4m+7)(2m+2)}\right]a_m$$

Letting m = 0, 1, etc., gives

$$a_1=-\frac{1}{14}a_0 \qquad a_2=-\frac{1}{(11)(4)}a_1=+\frac{1}{616}a_0 \qquad \text{etc.}$$

Therefore, since a_0 is arbitrary we can write the final form for $y_1(x)$ as

$$y_1(x)=x^{\frac{1}{4}}\left[1-\frac{1}{14}x+\frac{1}{616}x^2-\cdots\right]$$

A second linearly independent solution is obtained in a similar manner by setting $r = r_2 = -1/2$. For this case, after substitution of r_2 into the above balance relationships and some careful algebraic manipulation, we have

$$\sum_{m=0}\left\{\left[8m^2+10m+2\right]a_{m+1}+a_m\right\}x^{m+r+1}=\sum_{m=0}\left\{(4m+1)(2m+2)a_{m+1}+a_m\right\}x^{m+r+1}=0$$

Therefore, for $r = -1/2$, we have

$$a_{m+1}=-\left[\frac{1}{(4m+1)(2m+2)}\right]a_m$$

Letting m = 0, 1, etc., gives

$$a_1=-\frac{1}{2}a_0 \qquad a_2=-\frac{1}{20}a_1=+\frac{1}{40}a_0 \qquad \text{etc.}$$

Again, since a_0 is arbitrary, $y_2(x)$ becomes

$$y_2(x) = x^{-\frac{1}{2}}\left[1 - \frac{1}{2}x + \frac{1}{40}x^2 - \cdots\right]$$

and the final solution is simply a linear combination of the two linearly independent solutions, or

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

which, when written explicitly, gives

$$y(x) = c_1 x^{\frac{1}{4}}\left[1 - \frac{1}{14}x + \frac{1}{616}x^2 - \cdots\right] + c_2 x^{-\frac{1}{2}}\left[1 - \frac{1}{2}x + \frac{1}{40}x^2 - \cdots\right]$$

Example 7.3 -- Extended Power Series Solution (Dependent Solutions)

Problem Description:

Solve the following variable coefficient linear system:

$$\left(x^2-1\right)x^2y''-\left(x^2+1\right)xy'+\left(x^2+1\right)y=0$$

Problem Solution:

In standard form, the original linear ODE becomes

$$y''-\frac{\left(x^2+1\right)/\left(x^2-1\right)}{x}y'+\frac{\left(x^2+1\right)/\left(x^2-1\right)}{x^2}y=0$$

This form is similar to that observed in Example 7.2, thus we can use the extended power series method, with

$$y=\sum_m a_m x^{m+r}$$

and the derivative relations

$$y'=\sum_m a_m(m+r)x^{m+r-1} \qquad \text{and} \qquad y''=\sum_m a_m(m+r)(m+r-1)x^{m+r-2}$$

Substitution of these expressions into the base ODE gives

$$\sum_m a_m(m+r)(m+r-1)x^{m+r+2}-\sum_m a_m(m+r)(m+r-1)x^{m+r}$$
$$-\sum_m a_m(m+r)x^{m+r+2}-\sum_m a_m(m+r)x^{m+r}+\sum_m a_m x^{m+r+2}+\sum_m a_m x^{m+r}=0$$

Collecting like terms gives

$$\sum_m a_m\left[(m+r)(m+r-1)-(m+r)+1\right]x^{m+r+2}$$
$$+\sum_m a_m\left[-(m+r)(m+r-1)-(m+r)+1\right]x^{m+r}=0$$

and, with a little algebra, this becomes

$$\sum_m a_m\left[(m+r-1)(m+r-1)\right]x^{m+r+2}-\sum_m a_m\left[(m+r+1)(m+r-1)\right]x^{m+r}=0$$

Now, working on the second term, we let m = p+2 or p = m-2. Then the second term becomes

$$\sum_{p=-2} a_{p+2}\left[(p+r+3)(p+r+1)\right]x^{p+r+2}=a_0\left[(r+1)(r-1)\right]x^r+a_1\left[(r+2)r\right]x^{r+1}$$
$$+\sum_m a_{m+2}\left[(m+r+3)(m+r+1)\right]x^{m+r+2}$$

Therefore, for $a_0 \neq 0$, the roots of the indicial equation are simply $r_{1,2} = \pm 1$, and it should be noted that ***these do differ by an integer*** - thus, one is warned that dependent solutions may result. Also, from the requirement that the coefficient of the x^{r+1} term be zero, we have $a_1 = 0$, since $(r+2)r \neq 0$ for either r_1 or r_2.

Now to get the desired recurrence relation between a_{m+2} and a_m, we have

$$\sum_{m=0} \left[a_m (m+r-1)^2 - a_{m+2}(m+r+3)(m+r+1) \right] x^{m+r+2} = 0$$

and

$$a_{m+2} = \frac{(m+r-1)^2}{(m+r+3)(m+r+1)} a_m$$

Now for $r = r_1 = 1$, we have

$$a_{m+2} = \frac{m^2}{(m+4)(m+2)} a_m$$

Therefore, letting m = 0, 1, 2, etc., gives

$$a_2 = 0 a_0 = 0 \qquad a_3 = \frac{1}{(5)(3)} a_1 = 0 \qquad \text{etc.}$$

In fact, for all odd m, $a_m = 0$ since $a_1 = 0$, and for all even $m > 0$, $a_m = 0$ since $a_2 = 0$. Therefore, a_0 is the only non-zero term. This gives the simple result that

$$y_1(x) = a_0 x^r (1) = x$$

Performing similar manipulations for $r = r_2 = -1$, we have

$$a_{m+2} = \frac{(m-2)^2}{(m+2)(m)} a_m$$

but for m = 0, this term becomes undefined (i.e. $a_2 \to \infty$). This is clearly not allowed. Therefore, r = -1 is not a valid root. Actually, this behavior was not completely unexpected since r_1 and r_2 differ by an integer. In this case, the phenomenon observed here is not uncommon.

Note: If $r_1 - r_2 = N$ where N is an integer, then r_1, with $r_1 > r_2$, will always lead to a solution. The smaller root, r_2, might give both linearly independent solutions or it might lead nowhere -- that is, the smaller root either gives both solutions or none. In this case, $r_2 = -1$, was the smaller root and it did not give us any information about the solution to the ODE. Note that often it is best to check the smaller root first, since it will either give both solutions of nothing at all. If you check the largest root first, you will always get one solution. However, upon checking r_2, you may end up doing much of the same work again (if it leads to valid solutions). Thus, I recommend working with the smallest root first (even though I did not do that here…).

Now, to get a second linearly independent solution for our current problem, let's use the ***reduction of order method***. Here, this should be relatively straightforward, since $y_1(x)$ could be written as a simple closed form solution, where $y_1(x) = x$. In particular, to find $y_2(x)$, we let $y_2 = uy_1 = ux$. For this specific case, the relationships of interest are

$$y_2 = ux \qquad y_2' = u'x + u \qquad \text{and} \qquad y_2'' = u''x + u' + u' = u''x + 2u'$$

Substitution of these expressions into the defining ODE gives

$$\left(x^2 - 1\right)x^2\left[u''x + 2u'\right] - \left(x^2 + 1\right)x\left[u'x + u\right] + \left(x^2 + 1\right)ux = 0$$

Expanding and performing the indicated algebraic operations give

$$\left[x^5u'' + 2x^4u' - x^3u'' - 2x^2u'\right] - \left[x^4u' + x^3u + x^2u' + xu\right] + \left[x^3u + xu\right] = 0$$

or

$$x^3\left(x^2 - 1\right)u'' + x^2\left(x^2 - 3\right)u' = 0$$

This equation is separable and it can be simplified considerably using a partial fraction expansion technique, as follows:

Separating variables gives

$$\frac{u''}{u'} = \frac{3 - x^2}{x\left(x^2 - 1\right)} = \frac{3 - x^2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

where the A, B, and C constants are determined from

$$A = \left.\frac{3 - x^2}{(x+1)(x-1)}\right|_{x=0} = \frac{3}{-1} = -3$$

$$B = \left.\frac{3 - x^2}{x(x-1)}\right|_{x=-1} = \frac{2}{2} = 1$$

$$C = \left.\frac{3 - x^2}{x(x+1)}\right|_{x=1} = \frac{2}{2} = 1$$

Therefore, we have the simplified form

$$\frac{u''}{u'} = \frac{-3}{x} + \frac{1}{x+1} + \frac{1}{x-1}$$

This expression can be integrated directly, giving

$$\ln u' = -3\ln x + \ln\left(x + 1\right) + \ln\left(x - 1\right)$$

$$\ln u' = \ln\left(\frac{1}{x^3}\right) + \ln\left[\left(x+1\right)\left(x-1\right)\right] = \ln\left[\frac{x^2 - 1}{x^3}\right]$$

or

$$u' = \frac{x^2 - 1}{x^3}$$

One final integration gives

$$u(x) = \int \left[\frac{1}{x} - \frac{1}{x^3} \right] dx = \ln x + \frac{1}{2x^2}$$

Therefore, the second linearly independent solution becomes

$$y_2(x) = x \ln x + \frac{1}{2x}$$

Finally, the linear combination of the two independent solutions gives the desired general solution for this problem, or

$$y(x) = c_1 x + c_2 \left(x \ln x + \frac{1}{2x} \right)$$