Differential Equations (92.236)
Summary Information for Entire Course

First Order Equations

Various Forms of First Order Equations

$$\frac{dy}{dx} = F(x,y) \qquad \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \frac{dy}{dx} = f(x)$$

Separable Form

$$g(y)dy = f(x)dx$$

Notes:

If the system is ***homogeneous***, try the substitution $y = ux$ to convert the original ODE to separable form.

If there is an obvious ***algebraic combination*** of the form $ax + by + c$ that appears in every term containing x and y, letting $u = ax + by + c$ will convert the original equation to separable form.

First-Order Linear ODEs

$$y' + p(x)y = q(x) \qquad \text{with integrating factor} \quad g(x) = e^{\int p(x)dx}$$

Notes:

If the system is not linear, but can be written in the form of a ***Bernoulli Equation***,

$$y' + p(x)y = q(x)y^a$$

then letting $u(x) = y^{1-a}$ will convert the nonlinear ODE into a 1^{st} order linear system.

Exact Form

$$M(x,y)dx + N(x,y)dy = 0$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \qquad \text{exact if} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Notes:

If the system is not exact, multiplication by an ***integrating factor*** will make it exact (by definition). If the integrating factor is only a function of x, then it satisfies the equation

$$\frac{1}{g}\frac{dg}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \qquad \text{where } g(x) \rightarrow \text{integrating factor}$$

Finding a suitable integrating factor is not possible if the RHS is a function of both x and y.

Numerical Solution of IVPs

General Form for IVP

$$\frac{dy}{dx} = f(x,y) \qquad \text{with} \quad y(x_o) = y_o$$

Various One Step Algorithms (fixed step size, h)

	Euler $\varepsilon \approx O(h)$	**Improved Euler** $\varepsilon \approx O(h^2)$	**4th Order Runge Kutta** $\varepsilon \approx O(h^4)$
Evaluate function	$k_{1i} = hf(x_i, y_i)$	$k_{1i} = hf(x_i, y_i)$	$k_{1i} = hf(x_i, y_i)$
		$k_{2i} = hf(x_i + h, y_i + k_{1i})$	$k_{2i} = hf(x_i + \frac{h}{2}, y_i + \frac{k_{1i}}{2})$
			$k_{3i} = hf(x_i + \frac{h}{2}, y_i + \frac{k_{2i}}{2})$
			$k_{4i} = hf(x_i + h, y_i + k_{3i})$
Increment y	$y_{i+1} = y_i + k_{1i}$	$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)_i$	$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)_i$
Increment x	$x_{i+1} = x_i + h$	$x_{i+1} = x_i + h$	$x_{i+1} = x_i + h$

Notes:

The above methods generalize easily for systems of ODEs by using a vector notation.

The proper selection of h is quite difficult, especially for systems of ODEs. To resolve this problem, most modern ODE solvers automate the selection of the step size as the calculation proceeds to maintain a user specified error criterion. This feature is referred to as ***Adaptive Step Control*** and it requires a built-in feature to estimate the error associated with a given step size. This error estimate is usually obtained in one of two ways:

1. simultaneous calculation with two step sizes (h and 2h, for example)
2. comparison of low and high order approximations (i.e. a predictor-corrector method)

In general, h should not be too small because the number of steps will be large which leads to increased round-off error and computational costs. The step size should also not be too large because truncation error will be large. Adaptive step control represents a perfect compromise.

Autonomous Systems

An **autonomous system** is one in which the independent variable does not appear explicitly. For the case of a 1st order ODE, the general relationship, dy/dx = f(x,y), reduces to dy/dx = f(y) for autonomous systems. For such systems, the **critical points** are defined as the roots of the equation f(y) = 0. These roots are the values of y at which the derivative, dy/dx, is zero. At the critical point y = c, the ODE has the solution y(x) = c (a constant), and this solution curve is an equilibrium solution for the system. The sign of dy/dx in the vicinity of the critical or equilibrium points can give a lot of insight into the behavior of y(x) in the region near y(x) = c and as x becomes large.

Higher Order Linear ODEs

Some Terminology

n^{th} order linear ODE	$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$
2^{nd} order linear ODE	$y'' + p_1(x)y' + p_0(x)y = f(x)$
constant coeff. 2^{nd} order linear ODE	$y'' + ay' + by = f(x)$

A ***General Solution*** is given as the sum of the ***homogeneous*** and ***particular solutions***,

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution to the associated homogeneous equation [with f(x) = 0] and $y_p(x)$ is a particular solution to the full ODE including the external forcing function.

The ***Homogeneous Solution*** is a linear combination of n linearly independent individual solutions to the associated homogeneous ODE. It contains n arbitrary constants.

Linear Independence can be checked by computing the ***Wronskian*** of the individual solutions. If $W \neq 0$, then the individual solutions are linearly independent. For a 2^{nd} order system, the Wronskian is given as, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

The ***Particular Solution*** does not have any unknown constants. It can be determined by either the ***Method of Undetermined Coefficients*** or by the ***Variation of Parameters Method***.

A ***Unique Solution*** requires that n unique conditions be specified, and **these MUST be imposed upon the General Solution**.

IVP - There is only one boundary point location, x_o, which is usually referred to as the initial point. With n initial conditions at x_o, one can define the n constants uniquely. Thus, the solution to an IVP is unique (for continuous functions).

BVP - Two or more boundary points are specified. With n boundary conditions, one can at least define n-1 constants, with the final constant sometimes representing an arbitrary normalization (the difference here is specific to ***eigenvalue*** versus ***fixed-source*** problems).

Homogeneous Equations with Constant Coefficients (assume $y_h(x) = e^{rx}$)

With an assumed solution of the form $z(x) = e^{rx}$ for an n^{th} order linear ODE, one obtains the n^{th} order ***Characteristic Equation***: $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$

The n roots of the characteristic equation give the values of r that satisfy the original ODE. The roots of the n^{th} order polynomial equation can be classified as being real and distinct, real but repeated, complex conjugate pairs, or some combination of these. For a 2^{nd} order system, the roots of the characteristic equation, $ar^2 + br + c = 0$, can be written as $r_{1,2} = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

Real Distinct Roots: $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Complex Conjugate Pairs: $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ or $y_h(x) = e^{\alpha x}\left(a_1 \cos\beta x + a_2 \sin\beta x\right)$

where we used Euler's formula, $e^{\pm j\beta x} = \cos\beta x \pm j\sin\beta x$, and the roots written as $r_{1,2} = \alpha \pm j\beta$.

Repeated Roots: $y_h(x) = (c_1 + c_2 x)e^{rx}$

In this case, the root is repeated and only one independent solution is obtained. The second linearly independent solution is found using the ***Variation of Parameters*** method where we let $y_2(x) = u(x)y_1(x)$. For linear constant coefficient systems, u(x) is simply x to some power.

Higher Order Linear ODEs (cont.)

Finding Particular Solutions

In the ***Method of Undetermined Coefficients***, one assumes a solution, $y_p(x)$, that is of the same form as the forcing function **and** all its derivatives. For linear constant coefficient systems, if the choice of $y_p(x)$ via the above guideline is not linearly independent from the homogeneous solution, one simply multiplies the portion of the original choice associated with the homogeneous solution by x^m, where m is related to the multiplicity of the solution [use the lowest value of m that gives a linearly independent choice for $y_p(x)$]. Upon substitution into the original ODE, one equates coefficients of like terms to determine all the "undetermined coefficients" in the assumed solution.

The **Reduction of Order** method (sometimes called the **Variation of Parameter** method), where we choose $y_2(x) = u(x)y_1(x)$ with $y_1(x)$ being a known solution, is the formal technique used to develop the above procedure for repeated roots for constant coefficient systems.

For the **Variation of Parameters** method (for 2nd order systems), one chooses $y_p(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are to be determined and y_1 and y_2 are known solutions to the homogeneous equation. Substitution into the original ODE and an additional constraint (because we desire two unknown functions) gives the following system:

Given: $$y'' + p(x)y' + q(x)y = f(x)$$

Solve: $$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

With the solution of u_1' and u_2', a final integration step gives the desired $u_1(x)$ and $u_2(x)$ functions needed to generate the particular solution, $y_p(x)$.

Standard form for damped and undamped systems

The dynamic solution to second-order (or higher) linear constant coefficient systems often has periodic solutions of the form $y(t) = C_1\cos\omega t + C_2\sin\omega t$. This is often written in standard form as

$$y(t) = C\cos(\omega t - \alpha)$$

where C is the amplitude and α is known as the phase angle. These quantities are given in terms of the C_1 and C_2 coefficients, or

$$C = \sqrt{C_1^2 + C_2^2} \qquad \text{and} \qquad \alpha = \tan^{-1}(C_2/C_1) \qquad \text{(be careful with the quadrant for } \alpha\text{)}$$

Conversion of n^th^ order equations to n 1^st^ order equations

For most cases of interest, one simply makes the substitution

$$z_1 = y \qquad z_2 = y' = z_1' \qquad \cdots \qquad z_n = y^{(n-1)} = z_{n-1}'$$

These relationships, along with the original equation (with appropriate substitutions), give a system of 1st order equations of the form

$$\frac{d}{dx}\underline{z} = \underline{f}(x,\underline{z}) \qquad \text{with} \quad \underline{z}(x_o) = \underline{z}_o$$

This vector set of equations can be solved using the same numerical techniques discussed previously.

Laplace Transforms

Formal Definition of Laplace Transforms

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Inverse Laplace Transforms

$$f(t) = L^{-1}\{F(s)\}$$

The inversion process is usually obtained via a simple **Table Lookup** (see last page of these notes for some important Laplace Transform pairs and other useful relationships) after putting the function, F(s), into appropriate form (using partial fraction expansion techniques and some algebraic manipulation).

Solving Linear ODEs with Laplace Transforms

1. Take the Laplace Transform of every term in the original equation, being careful with the treatment of the initial condition terms associated with the derivative operators. This step converts the ODE into an algebraic equation.
2. Solve the algebraic equation for the dependent variable, Y(s).
3. Find the desired solution, y(t), by taking the inverse Laplace transform (i.e. $y(t) = L^{-1}\{Y(s)\}$)

Sum of Squares

The Laplace Transform inversion process often involves dealing with quadratic factors. When this occurs, the relationship,

$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2$$

can usually be used to assist in converting a ratio of polynomials in s into a form that is easier to invert.

Additional Goodies

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

examples:

$$\int x e^{ax} dx = x\left[\frac{e^{ax}}{a}\right] - \int\left[\frac{e^{ax}}{a}\right]dx = \frac{1}{a^2}e^{ax}\left[ax-1\right]$$

$$\int x\sin(ax)dx = x\left[\frac{-\cos(ax)}{a}\right] + \frac{1}{a}\int\cos(ax)dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax)$$

Additional Common Integrals

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\cos(ax)dx = \frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax)$$

$$\int \cos^2(ax)dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right)$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right)$$

Trigonometric Identities

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)$$

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$$