Differential Equations (92.236)
Summary Information for Exam #2

First Order Equations

Various Forms of First Order Equations

$$\frac{dy}{dx} = F(x,y) \qquad \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \frac{dy}{dx} = f(x)$$

Separable Form

$$g(y)dy = f(x)dx$$

Notes:

If the system is ***homogeneous***, try the substitution $y = ux$ to convert the original ODE to separable form.

If there is an obvious ***algebraic combination*** of the form $ax + by + c$ that appears in every term containing x and y, letting $u = ax + by + c$ will convert the original equation to separable form.

First-Order Linear ODEs

$$y' + p(x)y = q(x) \qquad \text{with integrating factor } g(x) = e^{\int p(x)dx}$$

Notes:

If the system is not linear, but can be written in the form of a ***Bernoulli Equation***,

$$y' + p(x)y = q(x)y^a$$

then letting $u(x) = y^{1-a}$ will convert the nonlinear ODE into a 1st order linear system.

Exact Form

$$M(x,y)dx + N(x,y)dy = 0$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \qquad \text{exact if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Notes:

If the system is not exact, multiplication by an ***integrating factor*** will make it exact (by definition). If the integrating factor is only a function of x, then it satisfies the equation

$$\frac{1}{g}\frac{dg}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \qquad \text{where } g(x) \rightarrow \text{integrating factor}$$

Finding a suitable integrating factor is not possible if the RHS is a function of both x and y.

Numerical Solution of IVPs

General Form for IVP

$$\frac{dy}{dx} = f(x, y) \qquad \text{with} \quad y(x_o) = y_o$$

Various One Step Algorithms (fixed step size, h)

	Euler $\varepsilon \approx O(h)$	**Improved Euler** $\varepsilon \approx O(h^2)$	**4th Order Runge Kutta** $\varepsilon \approx O(h^4)$
Evaluate function	$k_{1i} = hf(x_i, y_i)$	$k_{1i} = hf(x_i, y_i)$	$k_{1i} = hf(x_i, y_i)$
		$k_{2i} = hf(x_i + h, y_i + k_{1i})$	$k_{2i} = hf(x_i + \frac{h}{2}, y_i + \frac{k_{1i}}{2})$
			$k_{3i} = hf(x_i + \frac{h}{2}, y_i + \frac{k_{2i}}{2})$
			$k_{4i} = hf(x_i + h, y_i + k_{3i})$
Increment y	$y_{i+1} = y_i + k_{1i}$	$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)_i$	$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)_i$
Increment x	$x_{i+1} = x_i + h$	$x_{i+1} = x_i + h$	$x_{i+1} = x_i + h$

Notes:

The above methods generalize easily for systems of ODEs by using a vector notation.

The proper selection of h is quite difficult, especially for systems of ODEs. To resolve this problem, most modern ODE solvers automate the selection of the step size as the calculation proceeds to maintain a user specified error criterion. This feature is referred to as ***Adaptive Step Control*** and it requires a built-in feature to estimate the error associated with a given step size. This error estimate is usually obtained in one of two ways:

1. simultaneous calculation with two step sizes (h and 2h, for example)
2. comparison of low and high order approximations (i.e. a predictor-corrector method)

In general, h should not be too small because the number of steps will be large which leads to increased round-off error and computational costs. The step size should also not be too large because truncation error will be large. Adaptive step control represents a perfect compromise.

Autonomous Systems

An **autonomous system** is one in which the independent variable does not appear explicitly. For the case of a 1st order ODE, the general relationship, dy/dx = f(x,y), reduces to dy/dx = f(y) for autonomous systems. For such systems, the **critical points** are defined as the roots of the equation f(y) = 0. These roots are the values of y at which the derivative, dy/dx, is zero. At the critical point y = c, the ODE has the solution y(x) = c (a constant), and this solution curve is an equilibrium solution for the system. The sign of dy/dx in the vicinity of the critical or equilibrium points can give a lot of insight into the behavior of y(x) in the region near y(x) = c and as x becomes large.

Higher Order Linear ODEs

Some Terminology

n^{th} order linear ODE	$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$
2^{nd} order linear ODE	$y'' + p_1(x)y' + p_0(x)y = f(x)$
constant coeff. 2^{nd} order linear ODE	$y'' + ay' + by = f(x)$

A ***General Solution*** is given as the sum of the ***homogeneous*** and ***particular solutions***,

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution to the associated homogeneous equation [with f(x) = 0] and $y_p(x)$ is a particular solution to the full ODE including the external forcing function.

The ***Homogeneous Solution*** is a linear combination of n linearly independent individual solutions to the associated homogeneous ODE. It contains n arbitrary constants.

Linear Independence can be checked by computing the ***Wronskian*** of the individual solutions. If $W \neq 0$, then the individual solutions are linearly independent. For a 2^{nd} order system, the Wronskian is given as, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

The ***Particular Solution*** does not have any unknown constants. It can be determined by either the ***Method of Undetermined Coefficients*** or by the ***Variation of Parameters Method***.

A ***Unique Solution*** requires that n unique conditions be specified, and **these MUST be imposed upon the General Solution**.

IVP - There is only one boundary point location, x_o, which is usually referred to as the initial point. With n initial conditions at x_o, one can define the n constants uniquely. Thus, the solution to an IVP is unique (for continuous functions).

BVP - Two or more boundary points are specified. With n boundary conditions, one can at least define n-1 constants, with the final constant sometimes representing an arbitrary normalization (the difference here is specific to ***eigenvalue*** versus ***fixed-source*** problems).

Homogeneous Equations with Constant Coefficients (assume $y_h(x) = e^{rx}$)

With an assumed solution of the form $z(x) = e^{rx}$ for an n^{th} order linear ODE, one obtains the n^{th} order ***Characteristic Equation***: $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$

The n roots of the characteristic equation give the values of r that satisfy the original ODE. The roots of the n^{th} order polynomial equation can be classified as being real and distinct, real but repeated, complex conjugate pairs, or some combination of these. For a 2^{nd} order system, the roots of the characteristic equation, $ar^2 + br + c = 0$, can be written as $r_{1,2} = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$.

Real Distinct Roots: $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Complex Conjugate Pairs: $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ or $y_h(x) = e^{\alpha x}(a_1 \cos\beta x + a_2 \sin\beta x)$

where we used Euler's formula, $e^{\pm j\beta x} = \cos\beta x \pm j\sin\beta x$, and the roots written as $r_{1,2} = \alpha \pm j\beta$.

Repeated Roots: $y_h(x) = (c_1 + c_2 x)e^{rx}$

In this case, the root is repeated and only one independent solution is obtained. The second linearly independent solution is found using the ***Variation of Parameters*** method where we let $y_2(x) = u(x)y_1(x)$. For linear constant coefficient systems, u(x) is simply x to some power.

Additional Goodies

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

examples:

$$\int x e^{ax} dx = x\left[\frac{e^{ax}}{a}\right] - \int \left[\frac{e^{ax}}{a}\right] dx = \frac{1}{a^2} e^{ax}[ax - 1]$$

$$\int x \sin(ax) dx = x\left[\frac{-\cos(ax)}{a}\right] + \frac{1}{a}\int \cos(ax) dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax)$$

Additional Common Integrals

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x\cos(ax) dx = \frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax)$$

$$\int \cos^2(ax) dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int e^{ax}\sin(bx) dx = \frac{e^{ax}}{a^2 + b^2}\left(a\sin(bx) - b\cos(bx)\right)$$

$$\int e^{ax}\cos(bx) dx = \frac{e^{ax}}{a^2 + b^2}\left(a\cos(bx) + b\sin(bx)\right)$$

Trigonometric Identities

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)$$

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$$