

Differential Equations (92.236)

Summary Information for Exam #1

First Order Equations

Various Forms of First Order Equations

$$\frac{dy}{dx} = F(x, y) \qquad \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \frac{dy}{dx} = f(x)$$

Separable Form

$$g(y)dy = f(x)dx$$

Notes:

If the system is **homogeneous**, try the substitution $y = ux$ to convert the original ODE to separable form.

If there is an obvious **algebraic combination** of the form $ax + by + c$ that appears in every term containing x and y , letting $u = ax + by + c$ will convert the original equation to separable form.

First-Order Linear ODEs

$$y' + p(x)y = q(x) \qquad \text{with integrating factor } g(x) = e^{\int p(x)dx}$$

Notes:

If the system is not linear, but can be written in the form of a **Bernoulli Equation**,

$$y' + p(x)y = q(x)y^a$$

then letting $u(x) = y^{1-a}$ will convert the nonlinear ODE into a 1st order linear system.

Exact Form

$$M(x, y)dx + N(x, y)dy = 0$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0 \qquad \text{exact if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Notes:

If the system is not exact, multiplication by an **integrating factor** will make it exact (by definition). If the integrating factor is only a function of x , then it satisfies the equation

$$\frac{1}{g} \frac{dg}{dx} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \qquad \text{where } g(x) \rightarrow \text{integrating factor}$$

Finding a suitable integrating factor is not possible if the RHS is a function of both x and y .

Additional Goodies

Integration by Parts

$$\int u dv = uv - \int v du$$

examples: $\int x e^{ax} dx = x \left[\frac{e^{ax}}{a} \right] - \int \left[\frac{e^{ax}}{a} \right] dx = \frac{1}{a^2} e^{ax} [ax - 1]$

$$\int x \sin(ax) dx = x \left[\frac{-\cos(ax)}{a} \right] + \frac{1}{a} \int \cos(ax) dx = -\frac{x}{a} \cos(ax) + \frac{1}{a^2} \sin(ax)$$

Additional Common Integrals

$$\int \sin^2(ax) dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int x \cos(ax) dx = \frac{x}{a} \sin(ax) + \frac{1}{a^2} \cos(ax)$$

$$\int \cos^2(ax) dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \sin(bx) - b \cos(bx))$$

$$\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} (a \cos(bx) + b \sin(bx))$$

Trigonometric Identities

$$\sin x = \frac{1}{\csc x} \quad \cos x = \frac{1}{\sec x} \quad \tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = 2 \cos^2(x) - 1$$

$$\sin(x \pm y) = \sin(x) \cos(y) \pm \cos(x) \sin(y)$$

$$\cos(x \pm y) = \cos(x) \cos(y) \mp \sin(x) \sin(y)$$

$$\sin(x) + \sin(y) = 2 \sin\left(\frac{1}{2}(x+y)\right) \cos\left(\frac{1}{2}(x-y)\right)$$

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{1}{2}(x+y)\right) \cos\left(\frac{1}{2}(x-y)\right)$$

$$\sin(x) \sin(y) = \frac{1}{2} \cos(x-y) - \frac{1}{2} \cos(x+y)$$

$$\cos(x) \cos(y) = \frac{1}{2} \cos(x-y) + \frac{1}{2} \cos(x+y)$$

$$\sin(x) \cos(y) = \frac{1}{2} \sin(x+y) + \frac{1}{2} \sin(x-y)$$