92.236 Differential Equations
Exam #3 (Take Home) Spring 2006

Problem #1 (25 points)

a. Find the general solution to the following ODE: $y''' - 4y' = \sin x$

b. Find the general solution to the following ODE: $y''' + 4y' = \sin 2x$

c. Find the general solution to the following ODE: $4y'' + 4y' + y = x^{-2}e^{-x/2}$

d. Write a short Matlab file to find the **symbolic** solution of the above three ODEs. Include a listing of the Matlab m-file and a printout of the solutions as documentation for this part of the problem. Do your analytical solutions to Parts a-c agree exactly with the solutions from Matlab? Are they equivalent? Explain any differences…

Problem #2 (25 points)

Consider a forced mechanical system defined by $my'' + cy' + ky = f(t)$, with coefficients m = 1, k = 9.04, and c = 0.4 (with appropriate units). Assume that the system is initially at rest with y(0) = 0 m and y′(0) = 0 m/s.

a. Plot the **analytical** and **numerical** solutions for y(t) over the range 0 < t < 30 seconds, if the forcing function is given by $f(t) = 6\cos(3t)$.

b. Plot the **numerical** solution for y(t) over the same range, if the forcing function is given by $f(t) = 6e^{-t/5}\cos(3t)$.

Note that, even if a problem can be solved analytically, it is often easier to simply use a robust numerical ODE solver. This problem illustrates this situation quite nicely. In particular, you should solve Part a both analytically (by hand) and numerically (use Matlab's **ode23** or **ode45** routines for the numerical calculations). Plot the analytical and numerical solutions on the same plot and show that you get identical solutions. Which technique was easier for you to implement?

Now, for Part b, the analytical solution is rather tedious to work out -- can you explain why it becomes so tedious? Thus, based on the confidence gained with the numerical solution for Part a, you can obtain the solution to Part b using only the numerical approach. How does this profile compare to Part a? Considering the input forcing functions, do things make sense physically? Is the overall behavior as expected? Explain…

Note: Make sure your plots for both Parts a and b are properly labeled. Also include a listing of the Matlab files and any supporting analysis that you may have to document your work for this problem.

Problem #3 (25 points)

Consider a mechanical system that is characterized by mass m, spring constant k, and damping coefficient c. A force balance on the system gives the linear IVP

$$my'' + cy' + ky = f(t) \quad \text{with} \quad y(0) = y_o \ \text{ and } \ y'(0) = v_o$$

where y(t) represents the deviation from equilibrium, f(t) is an applied force, and y_o and v_o represent the initial position and velocity of the mass, respectively.

Within the context of this system, answer the following questions. Show your work and explain the logic used to rationalize your solutions.

a. If the measured response of the unforced system with m = 10 kg is of the form

$$y(t) = 5e^{-2t}\cos\left(3t - \frac{\pi}{6}\right)$$

what are the numerical values for the damping coefficient and spring constant for this system? What are the units of c and k? Assume that time is measured in seconds and that position is given in meters.

b. Write the measured response in Part a in the form

$$y(t) = e^{-2t}\left(A_1\cos 3t + A_2\sin 3t\right)$$

That is, what are the values of A_1 and A_2?

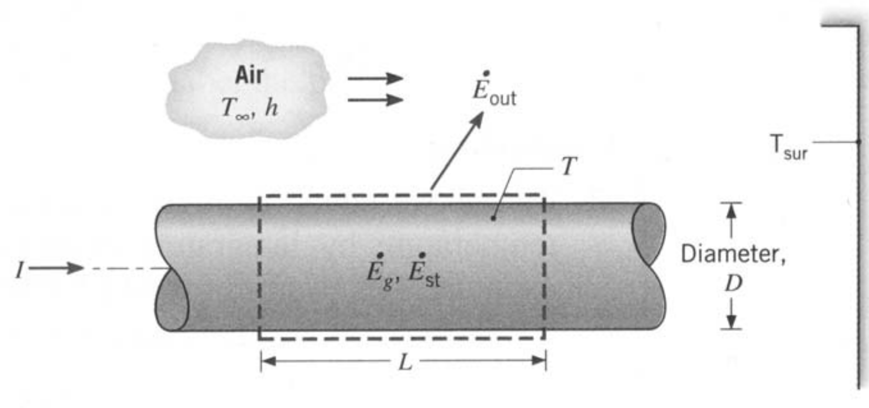
c. What initial conditions were required to give the response in Part a?

d. If the system is excited with a force that is given by $f(t) = 2\sin(10t)$, after a short transient period, what is the frequency of oscillation of the mass? Explain.

Problem #4 (25 points)

Consider a long conducting rod of diameter D and electrical resistance per unit length, R_L, as shown below. The rod is initially in thermal equilibrium with the ambient air and its surroundings. This situation changes, however, when an electrical current, I, is passed through the rod. When the current is turned on, the temperature should increase with time until the energy generated within the rod due to resistance heating equals the energy lost to the surroundings by convection and radiation heat transfer. Eventually, a new equilibrium temperature, consistent with the value of I, is reached and the system remains at this new equilibrium state until another change is made.

Considering the diagram given below, we can perform an energy balance on the indicated control volume as follows:



For a small-diameter rod, we can assume that the temperature throughout the rod is nearly uniform, with the temperature at any time, t, denoted as T(t). Thus, assuming a spatially uniform temperature at time t, an energy balance gives

$$\begin{bmatrix}\text{rate of energy}\\ \text{storage in CV}\end{bmatrix} = \begin{bmatrix}\text{energy generation rate}\\ \text{within CV by ohmic heating}\end{bmatrix} - \begin{bmatrix}\text{net energy flow rate}\\ \text{out of CV by}\\ \text{convection and radiation}\end{bmatrix}$$

or, in symbols,

$$\dot{E}_{st} = \dot{E}_g - \dot{E}_{out} \tag{1}$$

The energy within the control volume (CV) at any time can be written as

$$E_{CV} = \rho V u = \text{(mass/vol)(vol)(internal energy/mass)} = \text{energy in CV} \tag{2}$$

and, by definition of the internal energy, we have

$$du = c\, dT \tag{3}$$

where c is the specific heat, which quantifies the amount of increase in internal energy per unit mass that is observed for a one degree increase in temperature (with units of J/kg-C). Thus, assuming constant properties, we can write the rate of energy storage in the CV as

$$\dot{E}_{st} = \frac{d}{dt} E_{CV} = \rho c V \frac{dT}{dt} \tag{4}$$

For the energy generation term, we know from basic electricity that ohmic heating is given by

$$\dot{E}_g = P = I^2 R = I^2 R_L L \tag{5}$$

where, in this case, R_L is the resistance per unit length and the units in eqn. (5) are watts = $(\text{amps})^2$(ohms). Note that $R_L L$ is the total resistance of a rod of length L.

Now, assuming both convective and radiative losses to the surroundings, the energy loss rate can be written as

$$\dot{E}_{out} = hA\left(T - T_\infty\right) + \varepsilon\sigma A(T^4 - T_{sur}^4) \tag{6}$$

where A is the heat transfer surface area, T_∞ is the bulk fluid temperature, and T_{sur} represents the temperature of the surroundings for radiation heat transfer (note that absolute temperatures need to be used for radiation heat transfer problems). The remaining variables in the radiative term include the material's emissivity, ε, and the Stefan Boltzmann constant, $\sigma = 5.67\text{x}10^{-8}\ \text{W/m}^2\text{-K}^4$.

Finally, putting all these terms into the balance equation gives

$$\rho c V \frac{dT}{dt} = I^2 R_L L - hA\left(T - T_\infty\right) - \varepsilon\sigma A(T^4 - T_{sur}^4) \tag{7}$$

where, for this problem, $A = \pi DL$ and $V = \pi D^2 L/4$.

Thus, given the constant material properties (ρ, c, ε, and R_L), the geometry (D), and the environmental conditions (h, T_∞, and T_{sur}), we should be able to determine the T(t) that is associated with a step change in the electrical current, I, in the system. This requires solving the IVP defined in eqn. (7) subject to the initial condition, $T(0) = T_o$. For our analysis, we will assume that the initial rod temperature (with the current turned off) is in equilibrium with the air temperature, T_∞, and surrounding temperature, T_{sur}, such that $T_o = T_\infty = T_{sur}$.

Thus, for this problem, given the following material and geometry parameters, you need to solve the transient energy balance given in eqn. (7) for T(t) for three different step changes in the applied current:

bare copper wire:	D = 1 mm, $\rho = 8800\ \text{kg/m}^3$, c = 390 J/kg-K
	ε = 0.8, $R_L = 0.4$ ohm/m
environmental conditions:	$h = 100\ \text{W/m}^2\text{-K}$, $T_\infty = T_{sur} = 25$ C
initial condition:	$T(0) = T_o = 25$ C
current values to use:	I = 2, 5, and 10 amperes

For convenience in making comparisons, plot the three different solution curves (one T(t) for each value of I) on the same plot. Do your curves make sense? Are they as expected? How long after the current is turned on does it take to reach the new equilibrium? Is this a function of the value of I? Etc. Etc. In general, you should thoroughly discuss the results of this relatively simple heat transfer problem…

Note: This problem should be done using Matlab (or similar software). It is recommended that the **numerical** capability within Matlab be used for this problem. Also, when displaying your results, be sure that your plots are properly labeled. Also be sure you clearly discuss and interpret all of the results of your simulations! Finally, for completeness, be sure to include listings of all the Matlab files and any supporting analysis that you may have.

A professional job is expected here -- this includes a brief, well-written discussion of your results, a carefully prepared and properly labeled plot, and a correct and appropriately commented set of Matlab programs. All these items are needed for full documentation of this problem. Good Luck…