Differential Equations (92.236)

Exam #2 Spring 2006

Problem #1 (30 points)

Consider the initial value problem (IVP): $y' = y(x-1)$ with $y(0) = 5$

a. Solve the IVP for y(x) using analytical methods. From your solution, determine the value of y(x) for x = 0.2. That is, what is y(0.2)?

b. Using the **Euler** numerical integration technique, estimate the value of y(0.2) for the IVP given above. Use a step size h = 0.2. Compare the estimate of y(0.2) with the result from Part a and compute the percent error in the **Euler** estimate.

c. If the step size is reduced to h = 0.05, estimate the error at x = 0.2 for this case. Explain your logic.

Problem #2 (20 points)

The current, i(t), in a nonlinear electrical circuit is given by the solution to the following IVP:

$$2\frac{d}{dt}i + i^2 - 1 = 0 \qquad \text{with} \quad i(0) = i_o$$

a. Address the stability of the critical points and sketch the phase line for this first-order system.

b. Carefully sketch a series of expected solution curves, i(t), for various values of initial current, i_o. In particular, show the curves for i_o = -2.0, -0.5, 0.5, and 2.0 amps.

Problem #3 (30 points)

a. Find the general solution to the following ODEs:

$$(1)\ \ y'' - 4y = 0, \qquad (2)\ \ y'' - 6y' + 9y = 0, \qquad \text{and} \qquad (3)\ \ y'' + y = 0$$

b. Consider the inhomogeneous IVP given by

$$y''' - 4y'' - 5y' = 9 + 5x \qquad \text{with} \quad y(0) = 0, \quad y'(0) = 0, \quad \text{and} \quad y''(0) = 4$$

If the particular solution is given by $y_p(x) = -x - \frac{1}{2}x^2$, find the unique solution.

Problems #4 (20 points)

Any population contained in a finite living area with finite resources has a maximum size that it can sustain -- call this P_{max}. In such a system, a relatively simple but plausible statement of population growth is that the **absolute growth rate** (i.e. absolute birth rate - absolute death rate) is proportional to the difference between the limiting population and the population size at time t. This physical model can be written mathematically as

$$\frac{dP}{dt} = k\left(P_{max} - P\right) \qquad \text{with} \quad P(0) = P_o$$

with solution

$$\frac{P(t) - P_{max}}{P_o - P_{max}} = e^{-kt}$$

where k is the proportionality constant discussed above and P_o is the initial population size.

Assume that the finite living space is Planet Earth and that the particular species of interest is the human race. Current belief is that P_{max} is about 20 billion people (i.e. this is an estimate of the maximum population that the earth can sustain). The world population was about 4.5 billion people in 1980 and about 6.0 billion people in 1999. With these data points and the above simple result, estimate the number of people on Mother Earth in the year 2025.